A new div-curl result. Applications to the homogenization of elliptic systems and to the weak continuity of the Jacobian

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Abstract

In this paper a new div-curl result is established in an open set Ω of $\mathbb{R}^N$, $N \geq 2$, for the product of two sequences of vector-valued functions which are bounded respectively in $L^p(\Omega)^N$ and $L^q(\Omega)^N$, with $1/p + 1/q = 1 + 1/(N-1)$, and whose respectively divergence and curl are compact in suitable spaces. We also assume that the product converges weakly in $W^{-1,1}(\Omega)$. The key ingredient of the proof is a compactness result for bounded sequences in $W^{1,q}(\Omega)$, based on the imbedding of $W^{1,q}(S_{N-1})$ into $L^{p'}(S_{N-1})$ (S_{N-1} the unit sphere of $\mathbb{R}^N$) through a suitable selection of annuli on which the gradients are not too high, in the spirit of [26, 32]. The div-curl result is applied to the homogenization of equi-coercive systems whose coefficients are equi-bounded in $L^\rho(\Omega)$ for some $\rho > \frac{N-1}{2}$ if $N > 2$, or in $L^1(\Omega)$ if $N = 2$. It also allows us to prove a weak continuity result for the Jacobian for bounded sequences in $W^{1,N-1}(\Omega)$ satisfying an alternative assumption to the L^∞-strong estimate of [8]. Two examples show the sharpness of the results.

Keywords: div-curl, homogenization, elliptic systems, non equi-bounded coefficients, Γ-convergence, H-convergence, Jacobian, weak continuity.

Mathematics Subject Classification: 35B27, 74Q15

Contents

1 Introduction

In the early 1970s Murat and Tartar noticed that for any sequence σ_n weakly converging to σ in $L^p_{\rm loc}(\mathbb{R}^N)$, $N \geq 2$ and $p \in (1, \infty)$, and any sequence u_n converging weakly to u in $W^{1,p'}_{\rm loc}(\mathbb{R}^N)$ such that $\operatorname{div} \sigma_n$ converges strongly in $W^{-1,p}_{\rm loc}(\mathbb{R}^N)$, a simple integration by parts leads to the convergence

$$\sigma_n \cdot \nabla u_n \rightharpoonup \sigma \cdot \nabla u \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{1.1}$$

They extended this remark to the more general case where ∇u_n is replaced by any sequence η_n such that $\operatorname{curl} \eta_n$ is compact in $W^{-1,p'}_{\rm loc}(\mathbb{R}^N)$ (see [37]). The successful compensated compactness theory was born with a fruitful application to homogenization theory [36].

Actually, the elementary div-curl (1.1) contains hidden informations. Indeed, Coifman *et al.* proved that if $\operatorname{div} \sigma$ is in $W^{-1,s}_{\rm loc}(\mathbb{R}^N)$ with $s > p$, then $\sigma \cdot \nabla u$ belongs to the Hardy space $\mathscr{H}^1_{\rm loc}(\mathbb{R}^N)$. More recently, Conti *et al.* [21] obtained a new div-curl result relaxing the compensation conditions on $\operatorname{div} \sigma_n$ and $\operatorname{curl} \eta_n$ to the space $W^{-1,1}_{\rm loc}(\mathbb{R}^N)$, but assuming that the sequence $\sigma_n \cdot \eta_n$ is equi-integrable.

On the other hand, in the spirit of [36, 37] and using an appropriate Hodge decomposition of vector-valued fields, it was proved in [15] that, given a bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$, if $p, q \in [1, \infty)$ satisfy

$$p, q \geq 1, \quad \frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{N}, \tag{1.2}$$

and if σ_n, η_n satisfy the convergences

$$\sigma_n \rightharpoonup \sigma \begin{cases} L^p(\Omega)^N, & \text{if } p > 1 \\ \mathscr{M}(\Omega)^N *, & \text{if } p = 1, \end{cases} \qquad \eta_n \rightharpoonup \eta \begin{cases} L^q(\Omega)^N, & \text{if } q > 1 \\ \mathscr{M}(\Omega)^N *, & \text{if } q = 1, \end{cases} \tag{1.3}$$

and the compensation conditions

$$\operatorname{div} \sigma_n \to \operatorname{div} \sigma \begin{cases} W^{-1,q'}(\Omega)^N, & \text{if } q > 1 \\ L^N(\Omega)^N, & \text{if } q = 1, \end{cases} \qquad \operatorname{curl} \eta_n \to \operatorname{curl} \eta \begin{cases} W^{-1,p'}(\Omega)^N, & \text{if } p > 1 \\ L^N(\Omega)^N, & \text{if } p = 1, \end{cases} \tag{1.4}$$

then there exist two sequences x_j in Ω and c_j in $\mathbb{R}^N$ such that

$$\sigma_n \cdot \eta_n \rightharpoonup \sigma \cdot \eta + \sum_{j=1}^{\infty} \operatorname{div} (c_j \, \delta_{x_j}) \quad \text{in } \mathscr{D}'(\Omega). \tag{1.5}$$

In this paper we generalize the div-curl result of [37, 43, 15] assuming instead of (1.2) the inequality

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{N-1}. \tag{1.6}$$

The statement type is given by the following result which is refined in Theorem 2.1 (strict inequality in (1.6)) below:

Theorem 1.1. *Assume that* (1.6) *holds with the strict inequality. Consider two sequences* σ_n *in* $L^p(\Omega)^N$ *and* η_n *in* $L^{p'}(\Omega)^N$ *satisfying convergences* (1.3), (1.4) *with* $\sigma \in L^p(\Omega)^N$ *and* $\eta \in L^{p'}(\Omega)^N$, *and such that*

$$\sigma_n \cdot \eta_n \textit{ converges weakly in } W^{-1,1}(\Omega)^N. \tag{1.7}$$

Then, the weak limit of $\sigma_n \cdot \eta_n$ *is* $\sigma \cdot \eta$.

When equality holds in (1.6), Theorem 1.1 is also extended to Theorem 2.9 (case $p > 1$) and to Theorem 2.11 (case $p = 1$), under some equi-integrability assumption on $|\eta_n|$. Moreover, a counterexample to the div-curl result is given when this equi-integrability condition does not hold (see Proposition 2.15 below).

The proof of Theorem 1.1 differs notably from the ones of [37, 43, 15]. In fact, the improvement from the bound $1/N$ to the bound $1/(N-1)$ is connected to the imbedding, related to the unit sphere S_{N-1} of $\mathbb{R}^N$, of $W^{1,q}(S_{N-1})$ into $L^{p'}(S_{N-1})$, which is compact when inequality (1.6) is strict. Our approach is inspired by both

- De Giorgi's method [26] for matching boundary values in Γ-convergence, which consists in finding suitable annuli where the energy does not concentrate,
- Manfredi's method [32] for proving the continuity of a weakly monotone (*i.e.* satisfying a maximum principle) function in $W^{1,m}$, with $m > N-1$, which consists in selecting spheres on which the gradient of the function is not too high.[a]

Then, the key ingredient of the proof of Theorem 1.1 is given by the following result refined in Lemma 2.6 below:

Lemma 1.2. *Let $N \geq 2$, $0 < R_0 < R$, and $q > 1$. Consider a sequence u_n which converges weakly to u in $W^{1,q}\big(\{R_0 < |x| < R\}\big)$. Then, there exists a closed set $U_n \subset (R_0, R)$, whose measure is arbitrarily close to $R - R_0$, such that*

$$\begin{cases} \displaystyle\sup_{r\in U_n}\left(\int_{S_{N-1}} \big|u_n(ry) - u(ry)\big|^s\,ds(y)\right) \to 0, \quad 1 \leq s < q^*_{N-1} = \left(\tfrac{1}{q} - \tfrac{1}{N-1}\right)^{-1}, & \text{if } q < N-1 \\ \displaystyle\sup_{r\in U_n}\left(\int_{S_{N-1}} \big|u_n(ry) - u(ry)\big|^s\,ds(y)\right) \to 0, \quad 1 \leq s < \infty & \text{if } q = N-1 \\ \displaystyle\sup_{r\in U_n}\left(\sup_{y\in S_{N-1}} \big|u_n(ry) - u(ry)\big|\right) \to 0, & \text{if } q > N-1. \end{cases}$$

Lemma 1.2 means that one can select a n-dependent set U_n of annuli on which a strong estimate of $u_n - u$ holds. This set is built from not too high values of $|\nabla u_n|$ (see the definition (2.26) of U_n). Lemma 1.2 also extends to Lemma 2.13 in connection with Theorem 2.9 (critical case $s = q^*_{N-1}$), and to Lemma 2.14 (case $q = N-1$, with a uniform convergence result) in connection with Theorem 2.11, under a suitable equi-integrability assumption on $|\nabla u_n|$.

Beyond H-convergence for sequences of conductivity equations [36], which is historically linked to the classical div-curl lemma of [37], Tartar [43] extended its application field to various pde's including the hyperbolic equations. In the spirit of H-convergence, the div-curl approach was applied to linear elasticity in [25]. The seminal works [42, 36] on homogenization of elliptic problems are based on the boundedness (from below and above) of the sequences of coefficients involving in the equations. More recently, the boundedness assumption has been relaxed thanks to an appropriate extension of the div-curl lemma in conductivity [9, 11, 15], and in elasticity [10]. In these works the dimension is $N = 2$, and the sequences of coefficients are assumed to be uniformly bounded in L^1. The L^1-boundedness condition has been removed

[a]Manfredi's method was used in [13] to derive, thanks to the maximum principle, a uniform convergence result for sequences of solutions to elliptic equations with non equi-bounded coefficients. But of course this approach cannot be extended to elliptic systems.

in the setting of the homogenization of linear and nonlinear scalar problems [12, 7, 13] using the maximum principle in an essential way. Up to our knowledge, except the recent approach of [14] which is however based on a quite restrictive equi-integrability condition, the only available tool for deriving compactness results in the homogenization of sequences of systems with L^1-bounded coefficients remains the div-curl lemma. So, the linear elasticity result [10] shows that in dimension two the violation of the L^1-bound may induce second gradient terms in the homogenized equation. This anomalous behavior was previously observed in [38] in dimension three with a two-scale approach. In fact, the situation in three-dimensional linear elasticity is much more intricate since the closure set of equations is very large as shown in [18], while it is limited by the Beurling-Deny representation formula [4] in the conductivity case [17]. In view of the compactness result of [19] *versus* the nonlocal effects obtained in [27, 29, 3, 16, 17] and naturally connected with the Beurling-Deny formula by [34], the good assumption to avoid any loss of compactness in the homogenization process seems to be, at least in the scalar case and in any dimension, the equi-boundedness and the equi-integrability in L^1 of the sequences of coefficients.

In this context and as a by-product of the div-curl result of Theorem 1.1 and its extensions, we have the following homogenization result which is refined in Theorem 3.1 (with a Γ-convergence approach), and in Theorem 3.5 (with a H-convergence approach) below:

Theorem 1.3. *Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and let M be a positive integer. Consider a non-negative symmetric tensor-valued function $\mathbb{A}_n$ in $L^\infty(\Omega)^{(M\times N)^2}$ such that there exists a constant $\alpha > 0$ satisfying*

$$\alpha \int_\Omega |Dv|^2\,dx \leq \int_\Omega \mathbb{A}_n Dv : Dv\,dx, \quad \forall\, v \in H_0^1(\Omega)^M, \tag{1.8}$$

and such that

$$|\mathbb{A}_n| \text{ is bounded in } \quad \begin{cases} L^1(\Omega), & \text{if } N = 2 \\[2mm] L^\rho(\Omega), \quad \text{with } \rho > \dfrac{N-1}{2}, & \text{if } N > 2. \end{cases} \tag{1.9}$$

Then, there exist a subsequence of n, still denoted by n, and a non-negative symmetric tensor-valued $\mathbb{A} \in \mathscr{M}(\Omega)^{(M\times N)^2}$ if $N = 2$, or $\mathbb{A} \in L^\rho(\Omega)^{(M\times N)^2}$ if $N > 2$, satisfying (1.8), such that the following Γ-convergence for the $L^2(\Omega)^M$ strong topology holds

$$\begin{cases} \left(v \in H_0^1(\Omega)^M \mapsto \displaystyle\int_\Omega \mathbb{A}_n Dv : Dv\,dx\right) \xrightarrow{\Gamma} \left(v \in C_0^1(\Omega)^M \mapsto \displaystyle\int_\Omega \mathbb{A} Dv : Dv\,dx\right), & \text{if } N = 2 \\[2mm] \left(v \in H_0^1(\Omega)^M \mapsto \displaystyle\int_\Omega \mathbb{A}_n Dv : Dv\,dx\right) \xrightarrow{\Gamma} \left(v \in W_0^{1,\frac{2\rho}{\rho-1}}(\Omega)^M \mapsto \displaystyle\int_\Omega \mathbb{A} Dv : Dv\,dx\right), & \text{if } N > 2. \end{cases}$$

Note that in dimension three the result of Theorem 1.3 holds if the sequence $|\mathbb{A}_n|$ is bounded in some L^ρ space with $\rho > 1$. This condition is stronger than the equi-integrability of the coefficients, but is not so far from it. Alternatively, assuming that $\mathbb{A}_n$ is close in L^1-norm to an equi-coercive and equi-bounded sequence $\mathbb{B}^n$, we have obtained in [14] a similar compactness result by a quite different approach. Also note that the two-dimensional case of Theorem 1.3 includes the homogenization results of [11, 12, 10].

The classical div-curl lemma and more generally the compensated compactness has been successively used for weak continuity problems [36, 37, 45], and in particular for the weak continuity of the Jacobian in connection with the calculus of variations [33, 39, 1, 2, 22].

The divergence formulation of the Jacobian, denoted as Det, was originally established by Morrey [33], and leads to the classical weak continuity result (see, *e.g.*, [1, 23, 30, 28]): for any regular open bounded set Ω of $\mathbb{R}^N$, $N \geq 2$, and for any $s > \frac{N^2}{N+1}$,

$$u_n \rightharpoonup u \ \text{ in } W^{1,s}(\Omega)^N \quad \Rightarrow \quad \mathrm{Det}\,(Du_n) \rightharpoonup \mathrm{Det}\,(Du) \ \text{ in } \mathscr{D}'(\Omega). \tag{1.10}$$

Up to our knowledge, the most recent improvement of (1.10) is due to Brezis and Nguyen [8] who have obtained the weak continuity result

$$\left.\begin{array}{lll} u_n \rightharpoonup u & \text{in } W^{1,N-1}(\Omega)^N & \\ u_n \to u & \text{in } L^\infty(\Omega)^N, & \text{if } N=2 \\ u_n \to u & \text{in } BMO(\Omega)^N, & \text{if } N \geq 3 \end{array}\right\} \quad \Rightarrow \quad \mathrm{Det}\,(Du_n) \rightharpoonup \mathrm{Det}\,(Du) \ \text{ in } \mathscr{D}'(\Omega), \tag{1.11}$$

where the refinement in *BMO* is partly based on the div-curl approach of [20]. Actually, Brezis and Nguyen prove a delicate estimate (see [8], Theorem 1) which implies convergence (1.11).

Using the div-curl result of Theorem 2.11 we prove the alternative weak continuity convergence of the Jacobian in $W^{1,N-1}$ under different assumptions (see Theorem 3.8 below for a refined statement):

Theorem 1.4. *Let Ω be a bounded open set of $\mathbb{R}^N$, with $N \geq 2$. Consider a sequence of vector-valued functions $u_n = \left(u_n^1, \ldots, u_n^N\right)$ in $W^{1,N}(\Omega)^M$ satisfying*

$$u_n \rightharpoonup u = \left(u^1, \ldots, u^N\right) \quad \begin{cases} \text{in } W^{1,N-1}(\Omega)^N, & \text{if } N > 2 \\ \text{in } BV(\Omega)^N *, & \text{if } N = 2, \end{cases} \tag{1.12}$$

$$\mathrm{Det}\,(Du_n) \rightharpoonup \mu \quad \text{in } W^{-1,1}(\Omega). \tag{1.13}$$

Also assume that ∇u_n^1 is equi-integrable in the Lorentz space $L^{N-1,1}(\Omega)^N$.
Then, the limit distribution μ is given by the variational formulation

$$\langle \mu, \psi \rangle = -\int_\Omega \left(\sum_{j=1}^k \mathrm{cof}\,(Du)_{1j}\, \partial_j \psi\, u^1 \right) dx, \tag{1.14}$$

for a suitable dense set of radial fonctions ψ in $W_0^{1,\infty}(\Omega)$.

Example 3.10 below shows that the loss of equi-integrability for ∇u_n^1 may induce a concentration effect in the weak convergence of the Jacobian. This example also illustrates the sharpness of the weak continuity result of [8].

Notations

- M is a positive integer, and N is an integer ≥ 2.
- $(e_1, \ldots, e_N)$ denotes the canonical basis of $\mathbb{R}^N$, and $(f_1, \ldots, f_M)$ the one of $\mathbb{R}^M$.
- $:$ denotes the scalar product in $\mathbb{R}^{M\times N}$, *i.e.* $\xi : \eta = \mathrm{tr}\left(\xi^T \eta\right)$ for any $\xi, \eta \in \mathbb{R}^{M\times N}$.
- B_R denotes an open ball of $\mathbb{R}^N$ centered at the origin zero and of radius $R > 0$. The ball centered at the point x_0 and of radius R is denoted by $B(x_0, R)$.
- For $0 < R_0 < R$, $C(R_0, R)$ denotes the open crown $B_R \setminus \bar{B}_{R_0}$.

- S_{N-1} denotes the unit sphere of $\mathbb{R}^N$ for any integer $N \geq 2$.
- For any $p \in [1, \infty]$, $p' := \frac{p}{p-1} \in [1, \infty]$ denotes the conjugate exponent of p.
- For any $q \in [1, N)$, $q^*_N := \left(\frac{1}{q} - \frac{1}{N}\right)^{-1}$ denotes the critical Sobolev exponent in dimension N.
- $|E|$ denotes Lebesgue's measure of any measurable set $E \subset \mathbb{R}^N$. When E is a subset of a manifold of $\mathbb{R}^N$ of dimension $P \leq N$, $|E|$ is also used to denote the corresponding Hausdorff measure of order P.
- 1_E denotes the characteristic function of any set E.
- ∇u denotes the gradient of the scalar distribution $u : \mathbb{R}^N \to \mathbb{R}$.
- Du denotes the Jacobian matrix of the vector-valued distribution $u : \mathbb{R}^N \to \mathbb{R}^M$, *i.e.*

$$Du := \left[\frac{\partial u_i}{\partial x_j}\right]_{1 \leq i \leq M,\, 1 \leq j \leq N} \in \mathbb{R}^{M \times N}.$$

- div denotes the classical divergence operator acting on the vector-valued distributions.
- Div denotes the vector-valued differential operator taking the divergence of each row of a matrix-valued distribution,

$$\operatorname{Div} V := \left[\sum_{j=1}^{N} \frac{\partial V_{ij}}{\partial x_j}\right]_{1 \leq i \leq M}, \quad \text{for } V : \mathbb{R}^N \to \mathbb{R}^{M \times N}.$$

- curl denotes the classical curl operator acting on the vector-valued distributions.
- Curl denotes the vector-valued differential operator taking the curl of each row of a matrix-valued distribution,

$$\operatorname{Curl} V := \left[\frac{\partial V_{ij}}{\partial x_k} - \frac{\partial V_{ik}}{\partial x_j}\right]_{1 \leq i \leq M,\, 1 \leq j,k \leq N}, \quad \text{for } V : \mathbb{R}^N \to \mathbb{R}^{M \times N}.$$

- $\mathscr{M}(X)$ denotes the set of the bounded Radon measures on a locally compact set X.
- For a bounded open set Ω of $\mathbb{R}^N$, $W^{1,\infty}_0(\Omega)$ denotes the space of the functions in $W^{1,\infty}(\Omega)$ which are equal to 0 on $\partial\Omega$.
- $W^{-1,1}(\Omega)$ denotes the set composed of the divergences of functions in $L^1(\Omega)^N$. We can check that the dual of $W^{-1,1}(\Omega)$ is $W^{1,\infty}_0(\Omega)$ (using essentially the fact that the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$, and any vector-valued distribution which vanishes on the divergence free functions is a gradient). Hence, the weak convergence of μ_n to μ in $W^{-1,1}(\Omega)$ reads as

$$\langle \mu_n, \varphi \rangle \underset{n\to\infty}{\longrightarrow} \langle \mu, \varphi \rangle, \quad \forall\, \varphi \in W^{1,\infty}_0(\Omega). \tag{1.15}$$

  Note that the weak-$*$ convergence in $\mathscr{M}(\Omega)$ implies the weak convergence in $W^{-1,1}(\Omega)$.

2 The div-curl result

2.1 The case: $\frac{1}{p}+\frac{1}{q}<1+\frac{1}{N-1}$

We have the following div-curl result:

Theorem 2.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, with $N\geq 2$, and let $p,q\geq 1$ such that*

$$\frac{1}{p}+\frac{1}{q}<1+\frac{1}{N-1}. \tag{2.1}$$

Consider two sequences of matrix-valued functions σ_n and η_n such that

$$\exists\, s_n\in[p,q'],\quad \sigma_n\in L^{s_n}(\Omega)^{M\times N}\ \ \text{and}\ \ \eta_n\in L^{s_n'}(\Omega)^{M\times N}, \tag{2.2}$$

$$\sigma_n:\eta_n\rightharpoonup\mu\quad\text{weakly in } W^{-1,1}(\Omega). \tag{2.3}$$

Then, we have the following results according to the cases $p,q>1$, $q=1$ or $p=1$:

- *Assume that $p,q>1$, and that*

$$\begin{cases}\sigma_n\rightharpoonup\sigma & \text{in } L^p(\Omega)^{M\times N}\\ \eta_n\rightharpoonup\eta & \text{in } L^q(\Omega)^{M\times N},\end{cases} \tag{2.4}$$

$$\begin{cases}\operatorname{Div}\sigma_n\to\operatorname{Div}\sigma & \text{in } W^{-1,q'}(\Omega)^{M}\\ \operatorname{Curl}\eta_n\to\operatorname{Curl}\eta & \text{in } W^{-1,p'}(\Omega)^{M\times N\times N}.\end{cases} \tag{2.5}$$

  If the limits σ and η satisfy condition (2.2), then

$$\mu=\sigma:\eta. \tag{2.6}$$

  Otherwise, for any function u satisfying

$$u\in W^{1,q}(\Omega)^M\quad\text{and}\quad \eta-Du\in L^{p'}_{\mathrm{loc}}(\Omega)^{M\times N}, \tag{2.7}$$

  the limit μ satisfies the weak formulation

$$\begin{cases}\forall\, B(x_0,R)\Subset\Omega,\ \forall\,\varphi\in W^{1,\infty}(0,\infty),\ \text{with}\ \operatorname{supp}\varphi\subset[0,R],\\ \langle\mu,\psi\rangle=-\big\langle\operatorname{Div}\sigma,u\psi\big\rangle+\displaystyle\int_{B(x_0,R)}\sigma:\big[\eta\psi-D(u\psi)\big]\,dx,\\ \text{where}\ \psi(x):=\varphi(|x-x_0|).\end{cases} \tag{2.8}$$

- *Assume that $q=1$, and that*

$$\begin{cases}\sigma_n\rightharpoonup\sigma & \text{in } L^p(\Omega)^{M\times N}\\ \eta_n\overset{*}{\rightharpoonup}\eta & \text{in } \mathscr{M}(\Omega)^{M\times N},\end{cases} \tag{2.9}$$

$$\begin{cases}\operatorname{Div}\sigma_n\to\operatorname{Div}\sigma & \text{in } L^N(\Omega)^{M}\\ \operatorname{Curl}\eta_n\to\operatorname{Curl}\eta & \text{in } W^{-1,p'}(\Omega)^{M\times N\times N}.\end{cases} \tag{2.10}$$

  If the limits σ and η satisfy condition (2.2), then equality (2.6) holds.

  Otherwise, for any function u satisfying (2.7), the limit μ still satisfies the weak formulation (2.8).

- *Assume that $p = 1$, and that*

$$\begin{cases} \sigma_n \overset{*}{\rightharpoonup} \sigma & \text{in } \mathscr{M}(\Omega)^{M\times N} \\ \eta_n \rightharpoonup \eta & \text{in } L^q(\Omega)^{M\times N}, \end{cases} \tag{2.11}$$

$$\begin{cases} \operatorname{Div}\sigma_n \to \operatorname{Div}\sigma & \text{in } W^{-1,q'}(\Omega)^M \\ \operatorname{Curl}\eta_n \to \operatorname{Curl}\eta & \text{in } L^N(\Omega)^{M\times N\times N}. \end{cases} \tag{2.12}$$

If the limits σ and η satisfy condition (2.2)*, then equality* (2.6) *holds. Otherwise, for any function u satisfying*

$$u \in W^{1,q}(\Omega)^M \quad \text{and} \quad \eta - Du \in W^{1,N}_{\mathrm{loc}}(\Omega)^{M\times N}, \tag{2.13}$$

the limit μ satisfies the weak formulation

$$\begin{cases} \forall\, B(x_0,R) \Subset \Omega,\ \forall\, \varphi \in W^{1,\infty}(0,\infty),\ \text{with } \operatorname{supp}\varphi \subset [0,R],\ \text{such that} \\ \exists\, U \text{ closed set of } [0,R],\ \text{with } \left| \begin{array}{l} (r,y) \mapsto u(x_0+ry) \in C^0\big(U; W^{1,q}(S_{N-1})\big) \\ \varphi' \text{ is continuous on } U \text{ with support in } U, \end{array} \right. \\ \langle \mu, \psi\rangle = -\big\langle \operatorname{Div}\sigma, u\psi\big\rangle + \displaystyle\int_{B(x_0,R)} \sigma(dx) : \big[\eta\psi - D(u\psi)\big], \\ \text{where } \psi(x) := \varphi(|x-x_0|). \end{cases} \tag{2.14}$$

First of all, focus on the case $p, q > 1$:

Remark 2.2. First of all, in view of the weak formulation (2.8) note that

$$\sigma : \big[\eta\psi - D(u\psi)\big] = \sigma : (\eta - Du)\,\psi - (\sigma\nabla\psi)\cdot u.$$

Hence, since $\sigma : (\eta - Du)$ is in $L^1(\Omega)$ by (2.7), the last integral term of (2.8) has a sense if and only if the integral term

$$\int_{B(x_0,R)} (\sigma\nabla\psi)\cdot u\, dx,$$

has a sense. This needs radial test functions ψ and will be discussed in the general setting of Remark 2.4 below. However, observe that the set of functions ψ of the form

$$\psi(x) = \sum_{i=1}^{m} c_i\, \varphi_i(|x - x_i|)$$

such that for any $m \geq 1$ and $i \in \{1,\dots,m\}$, c_i is a real constant, $x_i \in \Omega$ and $\varphi_i \in W^{1,\infty}(0,\infty)$ with $\operatorname{supp}(\varphi_i) \subset [0,R_i]$, where $R_i > 0$ and $B(x_i,R_i) \Subset \Omega$, is dense in $W^{1,\infty}_0(\Omega)$. Therefore, the weak formulation (2.8) fully characterizes the distribution μ.

On the other hand, the existence of a function u satisfying (2.7) follows from the fact that $\operatorname{Curl}\eta$ belongs to $W^{-1,p'}(\Omega)^{M\times N\times N}$ (see, *e.g.*, [15], Proposition 2.5). Note that (2.8) does not depend actually on the choice of the function u satisfying (2.7). Indeed, let u and $\tilde{u}$ be two functions satisfying (2.7). Since $u - \tilde{u} \in W^{1,q}(\Omega)^N \cap W^{1,p'}_{\mathrm{loc}}(\Omega)^N$, we have

$$-\operatorname{Div}\sigma\cdot(u-\tilde{u}) - \sigma : D(u-\tilde{u}) + \operatorname{div}\big(\sigma^T(u-\tilde{u})\big) = 0 \quad \text{in } \Omega,$$

which implies that the right-hand side of (2.8) is equal to zero with $u - \tilde{u}$ instead of u.

Remark 2.3. It is clear that Theorem 2.1 implies the classical div-curl result of [37], [43], *i.e.* assuming that for $p \in (1,\infty)$,

$$\begin{cases} \sigma_n \rightharpoonup \sigma & \text{in } L^p(\Omega)^{M\times N} \\ \eta_n \rightharpoonup \eta & \text{in } L^{p'}(\Omega)^{M\times N}, \end{cases} \qquad \begin{cases} \operatorname{Div}\sigma_n \to \operatorname{Div}\sigma & \text{in } W^{-1,p}(\Omega)^M \\ \operatorname{Curl}\eta_n \to \operatorname{Curl}\eta & \text{in } W^{-1,p'}(\Omega)^{M\times N\times N}. \end{cases}$$

then the following convergence holds true

$$\sigma_n : \eta_n \overset{*}{\rightharpoonup} \sigma : \eta \quad \text{in } \mathscr{M}(\Omega).$$

We can also compare our result with the div-curl result of [15] based on the convergences (2.4) and (2.5) together with condition

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{N}. \tag{2.15}$$

First, by Proposition 2.5 of [15] there exists a matrix-valued function ζ such that

$$\zeta \in L^p(\Omega)^{M\times N}, \quad \operatorname{Div}\zeta = 0 \text{ in } \Omega, \quad \sigma - \zeta \in L^{q'}_{\mathrm{loc}}(\Omega)^{M\times N}. \tag{2.16}$$

Then, in the case $p, q > 1$ (but the other cases are similar), inequality (2.15) combined with the Sobolev imbedding $W^{1,q}(\Omega) \hookrightarrow L^{q^*_N}(\Omega)$ implies that if the functions u and ζ satisfy (2.7) and (2.16), then $\zeta^T u$ is in $L^1_{\mathrm{loc}}(\Omega)^N$. Therefore, using that ζ is divergence free, the limit formulation (2.8) can be written

$$\mu = \sigma : (\eta - Du) + (\sigma - \zeta) : Du + \operatorname{div}(\zeta^T u) \quad \text{in } \mathscr{D}'(\Omega), \tag{2.17}$$

which is the weak formulation for $\sigma : \eta$ according to Proposition 2.5 of [15]. However, Theorem 2.3 of [15] shows for sequences σ_n and η_n satisfying (2.2), (2.4), (2.5), the existence of two sequences x_j in Ω and c_j in $\mathbb{R}^N$ such that

$$\sigma_n : \eta_n \rightharpoonup \mu + \sum_{j=1}^{\infty} \operatorname{div}(c_j\, \delta_{x_j}) \quad \text{in } \mathscr{D}'(\Omega).$$

The reason of this apparent contradiction with equality (2.6) is that in Theorem 2.1 we have also assumed that $\sigma_n : \eta_n$ converges weakly in $W^{-1,1}(\Omega)$, while in [15] the convergence of $\sigma_n : \eta_n$ is obtained in the (larger) distributions space. It is easy to see that $\sigma_n : \eta_n$ in [15] is actually the divergence of a sequence which converges only in the weak-$*$ sense of the measures.

Remark 2.4. In view of (2.7) and (2.8) the regularity assumption (2.2) for σ_n and η_n, which holds in most situations, can be replaced in the case $p, q > 1$ by the more general conditions:

$$\sigma_n : \eta_n \in W^{-1,1}(\Omega), \tag{2.18}$$

and similarly to (2.8), for any $u_n \in W^{1,q}_{\mathrm{loc}}(\Omega)^M$ satisfying $\eta_n - Du_n \in L^{p'}_{\mathrm{loc}}(\Omega)^{M\times N}$, we have

$$\begin{cases} \forall\, B(x_0,R) \Subset \Omega,\ \forall\, \varphi \in W^{1,\infty}(0,\infty),\ \text{with } \operatorname{supp}\varphi \subset [0,R], \\ \big\langle \sigma_n : \eta_n, \psi \big\rangle = \ -\big\langle \operatorname{Div}\sigma_n, u_n\psi \big\rangle + \displaystyle\int_{B(x_0,R)} \sigma_n : \big[\eta_n\psi - D(u_n\psi)\big]\, dx \\ \qquad\qquad\qquad -\big\langle \operatorname{Div}\sigma_n, u_n\psi \big\rangle + \displaystyle\int_{B(x_0,R)} \big[\sigma_n : (\eta_n - Du_n)\,\psi - (\sigma_n\nabla\psi)\cdot u_n\big]\, dx, \\ \text{where } \psi(x) := \varphi(|x - x_0|). \end{cases} \tag{2.19}$$

So the distribution $\sigma_n : \eta_n$ is defined by the formula (2.19), and its extension to $W^{-1,1}(\Omega)$ is required through condition (2.18).

Then, we need to justify the integral term of (2.19)

$$\int_{B(x_0,R)} (\sigma_n \nabla \psi) \cdot u_n \, dx.$$

To this end, note that $u_n \in W^{1,q}_{\mathrm{loc}}(\Omega)^M$ implies that

$$\begin{array}{rccl} v_n : & (0,R) \times S_{N-1} & \to & \mathbb{R}^M \\ & (r,y) & \mapsto & u_n(x_0 + ry) \end{array}$$

is in $L^q_{r^{N-1}dr}(0,R;W^{1,q}(S_{N-1}))^M$, and thus by Sobolev's imbedding, in $L^q_{r^{N-1}dr}(0,R;L^{p'}(S_{N-1}))^M$ due to (2.1). Hence, at least for $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$ and

$$\operatorname{supp}(\varphi') \subset U_\lambda := \left\{ r \in (0,R) : \int_{\partial B(x_0,r)} |u_n|^{p'} \, ds(x) \leq \lambda \right\} \quad \text{for some } \lambda > 0, \tag{2.20}$$

we deduce that the right-hand side of (2.19) has a sense. But if (2.19) holds at least for φ satisfying (2.20), then using that $\sigma_n : \eta_n$ is in $W^{-1,1}(\Omega)$ the function

$$g_n : r \mapsto r^{N-1} \int_{S_{N-1}} \big(\sigma_n(x_0 + ry)\, y\big) \cdot u_n(x_0 + ry) \, ds(y)$$

satisfies, by (2.19) together with the definition of $W^{-1,1}$, the equality

$$\int_0^R \varphi' \, g_n \, dr = \int_0^R \varphi \, f_n \, dr + \int_0^R \varphi' \, h_n \, dr, \quad \text{where } f_n, h_n \in L^1(0,R),$$

which implies that for any $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$,

$$\int_0^R \varphi' \, 1_{U_\lambda} \, g_n \, dr = \int_0^R \left(\int_0^r \varphi' \, 1_{U_\lambda} \right) f_n \, dr + \int_0^R \varphi' \, 1_{U_\lambda} \, h_n \, dr.$$

This combined with $|U_\lambda| \to R$ as $\lambda \to \infty$, allows us to conclude that g_n is in $L^1(0,R)$. Therefore, the weak formulation (2.19) is actually satisfied for any $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$.

The same argument applies to the limit formulation (2.8). Moreover, following the first argument of Remark 2.2 the weak formulation (2.8) fully characterizes the distribution μ.

The case $q = 1$ is similar to the case $p, q > 1$. Now, focus on the case $p = 1$ which is more delicate concerning the sense of the weak formulation (2.14):

Remark 2.5. Assume that $p = 1$, and thus by (2.1) $q > N - 1$. With respect to the first term in the right-hand side of (2.14), since $\operatorname{Div}\sigma$ is in $W^{-1,q'}(\Omega)^M$, there exists a matrix-valued Radon measure ζ satisfying (see [15], Proposition 2.5)

$$\zeta \in \mathscr{M}(\Omega)^{M \times N}, \quad \operatorname{Div}\zeta = 0 \ \text{ in } \Omega, \quad \sigma - \zeta \in L^{q'}_{\mathrm{loc}}(\Omega)^{M \times N}. \tag{2.21}$$

Thanks to a result due to Bourgain, Brezis [5], the two first assertions of (2.21) imply that the measure ζ is actually in $W^{-1,N'}_{\mathrm{loc}}(\Omega)^{M \times N}$. Hence, it follows from (2.13) that

$$\sigma : (\eta - Du) = (\sigma - \zeta) : (\eta - Du) + \zeta : (\eta - Du) \in L^1_{\mathrm{loc}}(\Omega) + W^{-1,1}_{\mathrm{loc}}(\Omega), \tag{2.22}$$

which yields a sense to the integral term

$$\int_{B(x_0,R)} \sigma(dx) : (\eta - Du)\,\psi.$$

With respect to the last term in the right-hand side of (2.14), observe that the function $v : (0,R)\times S_{N-1} \to \mathbb{R}^M$ defined by $v(r,y) := u(x_0+ry)$ belongs to $L^q_{r^{N-1}dr}(0,R;W^{1,q}(S_{N-1}))^M$, and thus to $L^q_{r^{N-1}dr}(0,R;C^0(S_{N-1}))^M$ by Sobolev's imbedding due to $q > N-1$. Then, by Lusin's theorem, for any $\varepsilon > 0$, there exists of a closed set U satisfying the second line of (2.14) such that $|U| > R-\varepsilon$. For such a set U, the function v is in $C^0(U\times S_{N-1})$ (again by Sobolev's imbedding) and u is thus continuous on the closed set

$$K := \big\{x \in \bar\Omega : x = x_0 + ry,\ r \in U,\ y \in S_{N-1}\big\},$$

so that $\nabla\psi\otimes u$ can be extended to a continuous function in $\bar\Omega$. Therefore, the last term of (2.14), or equivalently,

$$\int_{B(x_0,R)} \big[\sigma : (\eta - Du)\,\psi - (\sigma\nabla\psi)\cdot u\big]\,dx,$$

in which

$$\int_{B(x_0,R)} (\sigma\nabla\psi)\cdot u\,dx = \int_K (\nabla\psi\otimes u) : d\sigma, \quad \text{where } \psi(x) := \varphi_U(|x-x_0|),$$

has a sense for any function φ_U satisfying the two first lines of (2.14). Moreover, since $|U|$ can be chosen arbitrarily close to R, any $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$, can be approximated for the weak-$*$ topology of $W^{1,\infty}(0,\infty)$ by a sequence of functions

$$\varphi_U(r) := \int_R^r \varphi'\,1_U\,ds, \quad \text{for } r \in [0,\infty).$$

But it is not clear that the sole condition $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$, is sufficient.

Finally, this combined with the first argument of Remark 2.2 implies that the weak formulation (2.14) fully characterizes the distribution μ.

2.2 Proof of Theorem 2.1

The key ingredient of the proof of Theorem 2.1 is the following compactness result:

Lemma 2.6. *Let $N \geq 2$, $0 < R_0 < R$, and $q \geq 1$. Consider a sequence u_n in $W^{1,q}(C(R_0,R))^M$ such that*

$$\begin{cases} u_n \rightharpoonup u & \text{in } W^{1,q}\big(C(R_0,R)\big)^M, \quad \text{if } q > 1 \\ u_n \overset{*}{\rightharpoonup} u & \text{in } BV\big(C(R_0,R)\big)^M, \quad \text{if } q = 1. \end{cases} \tag{2.23}$$

Define $v_n, v \in L^q(R_0,R;W^{1,q}(S_{N-1}))^M$, or $v \in L^1(R_0,R;BV(S_{N-1}))^M$ if $q = 1$, by

$$v_n(r,y) := u_n(ry), \quad v(r,y) := u(ry), \qquad \text{a.e. } (r,y) \in (R_0,R)\times S_{N-1}, \tag{2.24}$$

and the space X of functions in S_{N-1} by

$$X := \begin{cases} L^s(S_{N-1})^M, & \text{with } 1 \leq s < q^*_{N-1} = \left(\frac{1}{q} - \frac{1}{N-1}\right)^{-1}, \quad \text{if } q < N-1 \\ L^s(S_{N-1})^M, & \text{with } 1 \leq s < \infty, \quad \text{if } q = N-1 \\ C^0(S_{N-1})^M, & \quad \text{if } q > N-1. \end{cases} \tag{2.25}$$

Moreover, for any $\lambda > 0$ and any closed set U of $[R_0, R]$ such that $v \in C^0(U; W^{1,q}(S_{N-1}))^M$ if $q > 1$ or $v \in C^0(U; BV(S_{N-1}))^M$ if $q = 1$, define the subset U_n of U by

$$U_n := \left\{ r \in U : \int_{S_{N-1}} \big(|Du_n(ry)|^q + |Du(ry)|^q \big)\, ds(y) \le \lambda \right\}. \tag{2.26}$$

Then, we have

$$|U \setminus U_n| \le \frac{1}{\lambda\, R_0^{N-1}} \int_{\{|x| \in U\}} \big(|Du_n|^q + |Du|^q \big)\, dx, \tag{2.27}$$

$$\begin{cases} \|v_n - v\|_{C^0(U_n; X)} \to 0, & \text{if } q > 1 \\ \|v_n - v\|_{L^s(U_n; X)} \to 0, \quad \forall\, s \in [1, \infty), & \text{if } q = 1. \end{cases} \tag{2.28}$$

Proof. Property (2.27) is an immediate consequence of the definition (2.26) of U_n. Thus, we just need to prove (2.28).

On the one hand, since $W^{1,q}(S_{N-1})^M$ if $q > 1$, or $BV(S_{N-1})^M$ if $q = 1$, is compactly imbedded into X, we deduce from Lemma 5.1 of [31] that for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that

$$\begin{cases} \|w\|_X \le C_\delta \, \|w\|_{L^q(S_{N-1})^M} + \delta \, \|D_\tau w\|_{L^q(S_{N-1})^{M\times N}}, & \forall\, w \in W^{1,q}(S_{N-1})^M, \quad \text{if } q > 1, \\ \|w\|_X \le C_\delta \, \|w\|_{L^1(S_{N-1})^M} + \delta \, \|D_\tau w\|_{\mathscr{M}(S_{N-1})^{M\times N}}, & \forall\, w \in BV(S_{N-1})^M, \quad \text{if } q = 1, \end{cases}$$

where D_τ denotes the tangential derivative along the manifold S_{N-1}. Applying these inequalities to $(v_n - v)(r, \cdot)$, and taking into account the definition (2.26) of U_n, we get

$$\begin{cases} \|v_n - v\|_{C^0(U_n; X)} \le C_\delta \, \|v_n - v\|_{C^0(U_n; L^q(S_{N-1}))^M} + \delta \lambda^{\frac{1}{q}} & \text{if } q > 1 \\ \|v_n - v\|_{L^s(U_n; X)} \le C_\delta \, \|v_n - v\|_{L^s(U_n; L^1(S_{N-1}))^M} + \delta \lambda \, |U_n|^{\frac{1}{s}} & \text{if } q = 1. \end{cases} \tag{2.29}$$

On the other hand, the sequence $v_n - v$ is bounded in $L^q(R_0, R; W^{1,q}(S_{N-1}))^M$ and the sequence $\partial_r\big(v_n - v\big)$ is bounded in $L^q(R_0, R; L^q(S_{N-1}))^M$ if $q > 1$, or in $\mathscr{M}((R_0, R) \times S_{N-1})^M$ if $q = 1$. Hence, by a compactness result due to Simon [40] (Corollary 8 and Remark 10.1), the sequence $v_n - v$ converges strongly to 0 in

$$\begin{cases} C^0\big([R_0, R]; L^q(S_{N-1})\big)^M, & \text{if } q > 1 \\ L^m\big([R_0, R]; L^1(S_{N-1})\big)^M, \quad \forall\, m \in [1, \infty), & \text{if } q = 1, \end{cases} \tag{2.30}$$

which combined with (2.29) yields

$$\begin{cases} \displaystyle\limsup_{n\to\infty} \|v_n - v\|_{C^0(U_n; X)} \le \delta \lambda^{\frac{1}{q}}, & \text{if } q > 1 \\ \displaystyle\limsup_{n\to\infty} \|v_n - v\|_{L^s(U_n; X)} \le \delta \lambda \, R^{\frac{1}{s}}, & \text{if } q = 1. \end{cases}$$

Finally, the arbitrariness of $\delta > 0$ leads to (2.27). □

Let us start by the following preliminary remark which illuminates in particular the strategy of the proof of Theorem 2.1.

Remark 2.7. To fix ideas, assume that $p, q > 1$ with (2.1) (the other cases are similar). As observed in Remark 2.2, for $\sigma_n \in L^p(\Omega)^{M\times N}$ and $\eta_n \in L^q(\Omega)^{M\times N}$ such that $\operatorname{Curl}\eta_n$ is in $W^{-1,p'}(\Omega)^{M\times N\times N}$, there exists $u_n \in W^{1,q}(\Omega)^M$ such that $\eta_n - Du_n \in L^{p'}_{\rm loc}(\Omega)^{M\times N}$.

Then, for any $B(x_0, R) \Subset \Omega$ and for any $\varphi \in W^{1,\infty}(0,\infty)$ with

$$\operatorname{supp}\varphi \subset [0,R], \quad \operatorname{supp}(\varphi') \subset \left\{ r \in [0,R] : \int_{\partial B(x_0,r)} |\nabla u_n|^q \, ds(y) \le \lambda \right\} \text{ for some } \lambda > 0,$$

the integral

$$\int_\Omega \sigma_n : \big[\eta_n\psi - D(u_n\psi)\big] = \int_\Omega \big[\sigma_n : (\eta_n - Du_n)\,\psi - (\sigma_n\nabla\psi)\cdot u_n\big]\,dx,$$

where $\psi(x) := \varphi(|x - x_0|)$, is well defined. Defining V_n as the vector-space spanned by these functions ψ, we can then define the linear mapping $F_n : V_n \to \mathbb{R}$ by

$$F_n\psi := \int_\Omega \sigma_n : \big[\eta_n\psi - D(u_n\psi)\big]. \tag{2.31}$$

The proof of Theorem 2.1 essentially consists in constructing sequences ψ_n in V_n converging to a function ψ in $W^{1,\infty}(\Omega)$ weak-$*$ such that

$$F_n\psi_n \to F\psi.$$

But this does not prove the convergence of F_n to F in any topology because the spaces V_n vary with n. This is the reason to make assumption (2.3) in Theorem 2.1. However, this assumption can be simplified. Indeed, instead of assuming $\sigma_n : Du_n \in W^{-1,1}(\Omega)$, we can assume that

$$F_n \text{ defined by (2.31) can be extended to an element of } W^{-1,1}(\Omega), \tag{2.32}$$

which holds true for example if $\sigma_n^T u_n$ is in $L^1(\Omega)^N$, and then to define $\sigma_n : \eta_n$ in a relaxed way by the equality

$$\sigma_n : \eta_n := F_n. \tag{2.33}$$

Note that for σ_n, η_n smooth enough this equality holds, but F_n does not necessarily agree with the measurable function $\sigma_n : \eta_n$ which in general is not even in $L^1(\Omega)$. Then, also assuming

$$\sigma_n : \eta_n \rightharpoonup \mu \quad \text{in } W^{-1,1}(\Omega), \tag{2.34}$$

Theorem 2.1 shows that $\mu = \sigma : \eta$, where $\sigma : \eta$ is defined in a relaxed way similarly to $\sigma_n : \eta_n$.

The proof of Theorem 2.1 will use the following equi-integrability result for weakly convergent sequences in $W^{-1,1}(\Omega)$ and radial test functions:

Lemma 2.8. *Let $x_0 \in \Omega$ and $R > 0$ be such that $B(x_0,R) \subset \Omega$. Consider a sequence f_n in $L^1(\Omega)^N$ and a function f in $L^1(\Omega)^N$ such that* $\operatorname{div} f_n$ *converges weakly to* $\operatorname{div} f$ *in $W^{-1,1}(\Omega)$, and define h_n in $(0,R)$ by*

$$h_n(r) := \int_{\partial B(x_0,r)} f_n \cdot \frac{x - x_0}{|x - x_0|}\,ds, \quad \text{for } r \in (0,R). \tag{2.35}$$

Then, the sequence h_n is bounded and equi-integrable in $L^1(0,R)$.

Proof. It is equivalent to prove that h_n converges weakly in $L^1(0,R)$. For this purpose, consider $\phi \in L^\infty(0,R)$, and define $\varphi \in W^{1,\infty}(0,R)$ with $\varphi(R)=0$, by

$$\varphi(r) = \int_r^R \phi(t)\,dt \quad \text{for } r \in [0,R].$$

Then, we have

$$\int_0^R h_n\phi\,dr = -\int_{B(x_0,R)} f_n \cdot \frac{x-x_0}{|x-x_0|}\,\varphi'(|x-x_0|)\,dx = -\int_{B(x_0,R)} f_n \cdot \nabla\big[\varphi(|x-x_0|)\big]\,dx$$
$$= \big\langle \mathrm{div} f_n, \varphi(|x-x_0|)\big\rangle \underset{n\to\infty}{\longrightarrow} \big\langle \mathrm{div} f, \varphi(|x-x_0|)\big\rangle = \int_0^R h\phi\,dr,$$

where $h \in L^1(0,R)$ is defined replacing f_n by f in formula (2.35). Therefore, h_n converges weakly to h in $L^1(0,R)$. ☐

Proof of Theorem 2.1. First of all, if σ and η satisfy the regularity assumption (2.2), then the weak formulations (2.8) and (2.14) are reduced to $\mu = \sigma : \eta$. Indeed, in this case any function u satisfying (2.7) or (2.13) is in $W^{1,s'}(\Omega)^N$, so that

$$\mathrm{div}\,(\sigma^T u) = \mathrm{Div}\,(\sigma)\cdot u + \sigma : Du.$$

A simple integration by parts in (2.8) and (2.14) then yields $\mu = \sigma : \eta$.

Let us now treat the general case. From Proposition 2.5 of [15] we deduce the existence of functions u_n, u in $W^{1,q}(\Omega)^N$ satisfying

$$u_n \rightharpoonup u \quad \text{in} \begin{cases} W^{1,q}(\Omega)^M, & \text{if } q>1 \\ BV(\Omega)^M, & \text{if } q=1, \end{cases} \tag{2.36}$$

$$\eta_n - Du_n \to \eta - Du \quad \text{strongly in} \begin{cases} L^{p'}_{\mathrm{loc}}(\Omega)^{M\times N}, & \text{if } p>1 \\ W^{1,N}_{\mathrm{loc}}(\Omega)^{M\times N}, & \text{if } p=1. \end{cases} \tag{2.37}$$

Let be a closed ball of radius $R>0$ contained in Ω. Up to a translation we may assume the ball is centered at the origin. Define $v_n, v : (0,R)\times S_{N-1} \to \mathbb{R}^M$ by (2.24). For $R_0 \in (0,R)$ and for a closed set U of $[R_0,R]$ such that $v \in C^0(U; W^{1,q}(S_{N-1}))^M$, take a function $\varphi \in W^{1,\infty}(0,\infty)$ with $\mathrm{supp}\,\varphi \subset [0,R]$, $\mathrm{supp}\,(\varphi') \subset U$. Then, for a fixed $\lambda>0$, consider the set U_n defined by (2.26) and define the function $\varphi_n \in W^{1,\infty}(0,\infty)$ by

$$\varphi_n(r) := \int_R^r \varphi'\,1_{U_n}\,ds, \quad \text{for } r \in [0,\infty).$$

Also define the functions $\psi_n, \psi \in W^{1,\infty}_0(\Omega)$ by

$$\psi_n(x) := \varphi_n(|x|), \quad \psi(x) := \varphi(|x|), \quad \text{for } x \in \Omega.$$

According to Remark 2.7 our aim is to pass to the limit in $\big\langle \sigma_n : \eta_n, \psi_n\big\rangle$. We distinguish three cases:

• *The case* $p,q>1$. Using assumption (2.2) or the more general (2.19), combined with the first convergences of (2.5) and (2.37), we have

$$\begin{aligned} &\big\langle \sigma_n : \eta_n, \psi_n\big\rangle \\ &= -\big\langle \mathrm{Div}\,\sigma_n, u_n\psi_n\big\rangle + \int_\Omega \sigma_n : (\eta_n - Du_n)\,\psi_n\,dx - \int_{\{|x|\in U_n\}} (\sigma_n\nabla\psi_n)\cdot u_n\,dx \\ &= -\big\langle \mathrm{Div}\,\sigma, u\psi\big\rangle + \int_\Omega \sigma : (\eta - Du)\,\psi\,dx - \int_{\{|x|\in U_n\}} (\sigma_n\nabla\psi_n)\cdot u_n\,dx + o(1). \end{aligned} \tag{2.38}$$

On the one hand, to pass to the limit in the left-hand side of (2.38) we use the decomposition

$$\big\langle \sigma_n : \eta_n, \psi_n \big\rangle = \big\langle \sigma_n : \eta_n, \psi \big\rangle + \big\langle \sigma_n : \eta_n, \psi_n - \psi \big\rangle$$

where the first term converges clearly to $\langle \mu, \psi \rangle$ by (2.3). For the second term, by (2.3) there exist functions $f_n \in L^1(\Omega)^N$ satisfying $\operatorname{div} f_n = \sigma_n : \eta_n$. Thus, we have

$$\big|\big\langle \sigma_n : \eta_n, \psi_n - \psi \big\rangle\big| = \left| \int_\Omega f_n \cdot \nabla(\psi_n - \psi)\, dx \right| \leq C \int_{U \setminus U_n} |h_n|\, dr,$$

where h_n is defined by (2.35). Hence, by (2.27) we get that

$$\limsup_{n \to \infty} \big|\, \big\langle \sigma_n : \eta_n, \psi_n \big\rangle - \langle \mu, \psi \rangle \,\big| \leq C \sup_{\substack{m \in \mathbb{N} \\ |B| \leq c/\lambda}} \int_B |h_m|\, dr. \tag{2.39}$$

On the other hand, for the last term in (2.38), consider the functions v_n, v of (2.24) and define the functions $\xi_n, \xi : (0, R) \times S_{N-1} \to \mathbb{R}^{M \times N}$ by

$$\xi_n(r, y) := \sigma_n(ry), \quad \xi(r, y) := \sigma(ry), \quad \text{a.e. } (r, y) \in (0, R) \times S_{N-1}.$$

Then, we have

$$\begin{aligned} &\int_{\{|x| \in U_n\}} (\sigma_n \nabla \psi) \cdot u_n\, dx = \int_{U_n} \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot (v_n - v)\, ds(y)\, dr \\ &+ \int_U \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot v\, ds(y)\, dr - \int_{U \setminus U_n} \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot v\, ds(y)\, dr. \end{aligned} \tag{2.40}$$

Since ξ_n is bounded in $L^p(R_0, R; L^p(S_{N-1}))^{M \times N}$ and v_n satisfies the first convergence of (2.28) with $X := L^{p'}(S_{N-1})$ and $p' < q^*_{N-1}$ by (2.1), the first term in the right-hand side of (2.40) tends to zero. Moreover, since ξ_n converges weakly to ξ in $L^p(U; L^p(S_{N-1}))^{M \times N}$ and v is in $C^0(U; L^{p'}(S_{N-1}))^M$ by Sobolev's imbedding combined with $p' < q^*_{N-1}$, we have

$$\begin{aligned} \int_U \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot v\, ds(y)\, dr \;\; &\to \int_U \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi y) \cdot v\, ds(y)\, dr \\ &= \int_\Omega (\sigma \nabla \psi) \cdot u\, dx. \end{aligned}$$

The last term of (2.40) can be estimated thanks to Hölder's inequality by

$$\begin{aligned} &\left| \int_{U \setminus U_n} \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot v\, ds(y)\, dr \right| \\ &\leq R^{N-1}\, |U \setminus U_n|^{\frac{1}{p'}}\, \|\varphi'\|_{L^\infty(U)}\, \|\xi_n\|_{L^p(U; L^p(S_{N-1}))^{M \times N}}\, \|v\|_{C^0(U; L^{p'}(S_{N-1}))^M}, \end{aligned}$$

hence by (2.27)

$$\left| \int_{U \setminus U_n} \varphi'(r)\, r^{N-1} \int_{S_{N-1}} (\xi_n y) \cdot v\, ds(y)\, dr \right| \leq \frac{C}{\lambda^{\frac{1}{p'}}}. \tag{2.41}$$

Finally, combining (2.38), (2.39), (2.41) we obtain

$$\begin{aligned} &\left| \langle \mu, \psi \rangle + \big\langle \operatorname{Div} \sigma, u\psi \big\rangle - \int_\Omega \sigma : \big[\eta \psi - D(u\psi)\, dx \right| \\ &= \left| \langle \mu, \psi \rangle + \big\langle \operatorname{Div} \sigma, u\psi \big\rangle - \int_\Omega \sigma : (\eta - Du)\, \psi\, dx + \int_\Omega (\sigma \nabla \psi) \cdot u\, dx \right| \\ &\leq C \left(\sup_{\substack{m \in \mathbb{N} \\ |B| \leq c/\lambda}} \int_B |h_m|\, dr + \frac{1}{\lambda^{\frac{1}{p'}}} \right). \end{aligned}$$

Taking into account the equi-integrability of h_m given by Lemma 2.8 and the arbitrariness of $\lambda > 0$, we have just proved that the function u defined by (2.37) satisfies (2.8) for any $\varphi \in W^{1,\infty}(0,\infty)$ with $\operatorname{supp}\varphi \subset [0,R]$, and $\operatorname{supp}(\varphi')$ contained in a closed set U of $[R_0, R]$ such that v belongs to $C^0(U; W^{1,q}(S_{N-1}))^M$.

Finally, by Lusin's theorem the closed set U of $(0,R]$ can be chosen such that $R-|U|$ is arbitrary small. Hence, any function $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$, can be approximated for the weak-$*$ topology of $W^{1,\infty}(0,\infty)$ by a sequence of functions

$$\varphi_U(r) := \int_R^r \varphi'\, 1_U\, ds, \quad \text{for } r \in [0,\infty),$$

which satisfy $\operatorname{supp}(\varphi_U) \subset [0,R]$ and $\operatorname{supp}(\varphi_U') \subset U$. This combined with the density argument of Remark 2.4 (based on the fact that $\mu \in W^{-1,1}(\Omega)$) shows that the weak formulation (2.8) holds actually for any $\varphi \in W^{1,\infty}(0,\infty)$, with $\operatorname{supp}\varphi \subset [0,R]$. This concludes the proof of Theorem 2.1 in the case $p,q > 1$.

• *The case* $q = 1$. It is similar to the previous case using the first convergence of (2.37), and the second convergence of (2.36) combined with Sobolev's imbedding $BV(\Omega)^M \hookrightarrow L^{N'}(\Omega)^M$.

• *The case* $p = 1$. It is also similar to the first case. The only delicate point comes from the second term in the right-hand side of (2.38). In view of (2.21) and (2.22) we can write

$$\int_\Omega \sigma_n : (\eta_n - Du_n)\,\psi_n\, dx = \int_\Omega (\sigma_n - \zeta_n) : (\eta_n - Du_n)\,\psi_n\, dx + \int_\Omega \zeta_n : (\eta_n - Du_n)\,\psi_n\, dx, \quad (2.42)$$

where by virtue of Proposition 2.5 of [15] the measures ζ_n, ζ satisfy

$$\zeta_n \rightharpoonup \zeta \quad \text{in } \mathscr{M}(\Omega)^{M\times N}, \quad \operatorname{Div}\zeta_n = 0 \ \text{ in } \Omega, \quad \sigma_n - \zeta_n \to \sigma - \zeta \ \text{ strongly in } L^{q'}_{\mathrm{loc}}(\Omega)^{M\times N}. \quad (2.43)$$

By the second convergence of (2.37) and (2.43) the first term in the right-hand side of (2.42) clearly converges. Moreover, we can also pass to the limit in the second term of the right-hand side of (2.42), since the divergence free sequence ζ_n converges weakly in $W^{-1,N'}(\Omega)^{M\times N}$ thanks to the Bourgain, Brezis result [5], hence

$$\int_\Omega \sigma_n : (\eta_n - Du_n)\,\psi_n\, dx \to \int_\Omega \sigma : (\eta - Du)\,\psi\, dx.$$

Therefore, the proof of Theorem 2.1 is complete. □

2.3 The limit case: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{N-1}$

When inequality (2.1) becomes an equality, the imbedding $W^{1,q}(S_{N-1}) \hookrightarrow L^{p'}(S_{N-1})$ is no more compact, so Lemma 2.6 is useless. This lack of compactness can be overcome adding an equi-integrability assumption for the sequence η_n in Theorem 2.1. This is the aim of Theorem 2.9 in the case $p > 1$.

The case $p = 1$, and thus $q = N-1$, corresponds to the critical case for Sobolev's inequality: $W^{1,N-1}(S_{N-1})$ is continuously imbedded in $L^s(S_{N-1})$ for any $s \in [1,\infty)$, but if $N > 2$, it is not imbedded in $L^\infty(S_{N-1})$. To get over this difficulty we need to work with a space which is a little more regular than $L^{N-1}(\Omega)$. So, in Theorem 2.11 below $L^{N-1}(\Omega)$ is replaced by the Lorentz space $L^{N-1,1}(\Omega)$. It is known that the space of functions $u \in W^{1,N-1}(S_{N-1})$ the gradient of which belongs to $L^{N-1,1}(S_{N-1})$ is continuously imbedded in $C^0(S_{N-1})$ (see, *e.g.*, [44], Chap. 31). Moreover, for $N = 2$, $L^{1,1}(S_1)$ agrees with $L^1(S_1)$, so that we can extend Theorem 2.1 to the case $N = 2$, $p = q = 1$.

Theorem 2.9. *Let Ω be a bounded open set of $\mathbb{R}^N$, with $N \geq 2$, and let p, q be such that*

$$1 < p \leq N-1, \quad 1 \leq q < N-1, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{N-1}. \tag{2.44}$$

Consider two sequences of matrix-valued functions $\sigma_n \in L^p(\Omega)^{M\times N}$, $\eta_n \in L^q(\Omega)^{M\times N}$, satisfying (2.2), (2.3), (2.4), (2.5) *together with*

$$|\eta_n|^p \text{ equi-integrable in } L^1(\Omega). \tag{2.45}$$

Then the weak formulation (2.8) *holds true.*

In order to state the case $p = 1$, $q = N-1$, recall the definition of the Lorentz space $L^{p,1}(E)$:

Definition 2.10. Let E be a measurable set of $\mathbb{R}^N$. For a measurable function $f : E \to \mathbb{R}$, the non-increasing rearrangement $f^* : [0,\infty) \to \mathbb{R}$ of f is defined by

$$f^*(t) := \inf\big\{\lambda \geq 0 : \big|\{x \in E : |f(x)| > \lambda\}\big| \leq t\big\}. \tag{2.46}$$

Then, we define $L^{p,1}(E)$, $p > 1$, as the space of measurable functions $f : E \to \mathbb{R}$ such that

$$\|f\|_{L^{p,1}(E)} = \int_0^\infty t^{-\frac{1}{p'}} f^*(t)\, dt = \int_0^\infty \big|\{x \in E : |f(x)| > \lambda\}\big|^{\frac{1}{p}}\, d\lambda < \infty. \tag{2.47}$$

The space $L^{p,1}(E)$ is a Banach space equipped with the norm $\|\cdot\|_{L^{p,1}(E)}$.

Theorem 2.11. *Let Ω be a bounded open set of $\mathbb{R}^N$, with $N \geq 2$, and two sequences of matrix-valued functions σ_n and η_n satisfying* (2.2), (2.3),

$$\begin{cases} \sigma_n \overset{*}{\rightharpoonup} \sigma & \text{in } \mathscr{M}(\Omega)^{M\times N} \\ \eta_n \rightharpoonup \eta & \text{in } L^{N-1,1}(\Omega)^{M\times N}, \end{cases} \tag{2.48}$$

$$\begin{cases} \operatorname{Div}\sigma_n \to \operatorname{Div}\sigma & \text{in } W^{-1,(N-1)'}(\Omega)^M \\ \operatorname{Curl}\eta_n \to \operatorname{Curl}\eta & \text{in } L^N(\Omega)^{M\times N\times N}. \end{cases} \tag{2.49}$$

Also assume that the sequence η_n satisfies the equi-integrability condition

$$\forall\,\varepsilon > 0,\ \exists\,\delta > 0, \quad \|\eta_n\|_{L^{N-1,1}(E)^{M\times N}} \leq \varepsilon, \ \ \forall\, n \in \mathbb{N},\ \forall\, E \text{ measurable set of } \Omega,\ |E| < \delta. \tag{2.50}$$

Then, for any function u satisfying

$$u \in W^{1,N-1}(\Omega)^M, \quad Du \in L^{N-1,1}(\Omega)^{M\times N}, \quad \eta - Du \in W^{1,N}(\Omega)^{M\times N}, \tag{2.51}$$

the limit μ satisfies the weak formulation

$$\begin{cases} \forall\, B(x_0,R) \Subset \Omega,\ \forall\,\varphi \in W^{1,\infty}(0,\infty),\ \text{with } \operatorname{supp}\varphi \subset [0,R],\ \text{such that} \\ \exists\, U \text{ closed set of } [0,R],\ \text{with } u(x_0+ry) \in C^0(U; X^{1,N-1}(S_{N-1}))^M,\ \operatorname{supp}(\varphi') \subset U, \\ \langle \mu, \psi\rangle = -\big\langle \operatorname{Div}\sigma, u\psi\big\rangle + \displaystyle\int_{B(x_0,R)} \big[\sigma(dx) : (\eta - Du)\,\psi - (\sigma\nabla\psi)\cdot u\, dx\big], \\ \text{where } \psi(x) := \varphi(|x-x_0|), \end{cases} \tag{2.52}$$

and $X^{1,N-1}(S_{N-1})$ is the space defined by

$$X^{1,N-1}(S_{N-1}) := \left\{v \in W^{1,N-1}(S_{N-1}) : \nabla v \in L^{N-1,1}(S_{N-1})^N\right\}. \tag{2.53}$$

Remark 2.12. Let u be a function in $W^{1,N-1}(\Omega)^M$ such that $Du \in L^{N-1,1}(\Omega)^{M\times N}$, and let $v:(0,R)\times S_{N-1}\to\mathbb{R}^M$ be the function defined by $v(r,y):=u(x_0+ry)$, so that $\nabla_y v$ is the tangential part of ∇u on $\partial B(x_0,r)$. By Hölder's inequality we have for any $\lambda>0$,

$$\int_0^R r^{N-1}\left(\int_{S_{N-1}} 1_{\{|\nabla_y v|>\lambda\}}\,ds(y)\right)^{\frac{1}{N-1}} dr \le CR^{N'(N-2)}\left(\int_0^R\int_{S_{N-1}} 1_{\{|\nabla u(x_0+ry)|>\lambda\}}\,ds(y)\,r^{N-1}\,dr\right)^{\frac{1}{N-1}}$$
$$\le CR^{N'(N-2)}\Big(\big|\{x\in B(x_0,R):|\nabla u(x)|>\lambda\}\big|\Big)^{\frac{1}{N-1}}.$$

Hence, integrating the previous inequality with respect to $\lambda>0$ and using that $\nabla u\in L^{N-1,1}(\Omega)^N$, it follows that v is in $L^1_{r^{N-1}dr}(0,R;X^{1,N-1}(S_{N-1}))^M$, and thus in $L^1_{r^{N-1}dr}(0,R;C^0(S_{N-1}))^M$ since the Lorentz space $L^{N-1,1}(S_{N-1})$ is imbedded into $C^0(S_{N-1})$ (see [44], Chap. 31). Moreover, by Lusin's theorem, for any $\varepsilon>0$, there exists a closed set U satisfying the second line of (2.52) such that $|U|>R-\varepsilon$. Hence, for $\sigma\in\mathscr{M}(\Omega)^{M\times N}$, the integral term

$$\int_{B(x_0,R)}(\sigma\nabla\psi)\cdot u\,dx,\quad \text{where } \psi(x):=\varphi(|x-x_0|),$$

has a sense for any function φ satisfying the two first lines of (2.52). Therefore, we can conclude as in Remark 2.5 that the weak formulation (2.52) fully characterizes the distribution μ.

The proof of the two last theorems is similar to the one of Theorem 2.1 using Lemma 2.13 below in the case $p>1$, and Lemma 2.14 below in the case $p=1$, instead of Lemma 2.6. So we restrict ourselves to the proof of these two lemmas.

Lemma 2.13. *Let $N>2$, let $R_0,R>0$ be such that $R_0<R$, and let $q\in[1,N-1)$. Consider a sequence u_n in $W^{1,q}(C(R_0,R))$ which converges weakly to a function u in $W^{1,q}(C(R_0,R))$, and such that $|\nabla u_n|^q$ is equi-integrable in $L^1(\Omega)$. Consider $v_n,v\in L^q(R_0,R;W^{1,q}(S_{N-1}))$ defined by* (2.24).
Then, for any U subset of $[R_0,R]$ such that $v\in L^\infty(U;L^{q^*_{N-1}}(S_{N-1}))^M$, for any $\lambda,\varepsilon>0$, there exists a sequence $U_n\subset U$ satisfying

$$|U\setminus U_n|\le\frac{1}{R_0^{N-1}}\left(\frac{1}{\lambda}\int_{\{|x|\in U\}}\big(|\nabla u_n|^q+|\nabla u|^q\big)\,dx+\varepsilon\right),\tag{2.54}$$

$$\int_{S_{N-1}}\big(|\nabla u_n(ry)|^q+|\nabla u(ry)|^q\big)\,ds(y)<\lambda,\qquad a.e.\ r\in U_n,\tag{2.55}$$

$$\|v_n-v\|_{L^\infty(U_n;L^{q^*_{N-1}}(S_{N-1}))}\to 0.\tag{2.56}$$

Proof. Since $W^{1,p}(C(R_0,R))$ is compactly imbedded in $L^1(\partial B(0,r))$ for any $r\in[R_0,R]$, the sequence $v_n(r,.)$ converges to $v(r,.)$ in $L^1(S_{N-1})^M$ for any $r\in[R_0,R]$. Also using that

$$\|v_n(r_1,\cdot)-v_n(r_2,\cdot)\|_{L^1(S_{N-1})}\le C\int_{\{r_1<|x|<r_2\}}|\nabla u_n|\,dx,\quad \forall\, r_1,r_2 \text{ with } R_0<r_1<r_2<R.$$

and the equi-integrability of $|\nabla u_n|$ in $L^1(R_0,R)$, we easily conclude that v_n converges to v in $C^0([R_0,R];L^1(S_{N-1}))^M$.

Now, take $\varepsilon > 0$. By the equi-integrability of $|Du_n|^q$, for any $k \in \mathbb{N}$, there exists $\delta_k > 0$ such that for any measurable set $B \subset C(R_0, R)$ with $|B| < \delta_k$, we have

$$\int_B \Lambda_n \, dx < \frac{\varepsilon^2}{2^{2k}}, \quad \forall\, n \in \mathbb{N}, \qquad \text{where } \Lambda_n := |\nabla u_n|^q + |\nabla u|^q. \tag{2.57}$$

Let $\phi : (0, \infty) \to \mathbb{R}$ the function defined by

$$\phi(h) := \big|B(e_1, h) \cap S_{N-1}\big|, \quad \text{for } h > 0, \tag{2.58}$$

and let $h_k > 0$ be such that

$$\phi(h_k)\, \frac{R^N - R_0^N}{N} < \delta_k. \tag{2.59}$$

Then, for a.e. $r \in (R_0, R)$ and any $n, k \in \mathbb{N}$, denote

$$T_{n,k}(r) := \sup_{z \in S_{N-1}} \int_{B(z,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y).$$

We will prove that the set

$$E_{n,k} := \left\{ r \in (R_0, R) : T_{n,k}(r) > \frac{\varepsilon}{2^k} \right\}, \quad \text{for } k, n \in \mathbb{N}, \tag{2.60}$$

satisfies

$$|E_{n,k}| < \frac{\varepsilon}{2^k R_0^{N-1}}, \quad \forall\, k, n \in \mathbb{N}. \tag{2.61}$$

To this end, for fixed $k, n \in \mathbb{N}$, consider for a.e. $r \in (0, R)$,

$$F(r) := \left\{ z \in S_{N-1} : \int_{B(z,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y) = \sup_{x \in S_{N-1}} \int_{B(x,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y) \right\}.$$

Then, F is a multifunction valued on closed sets. Let us also prove that it is measurable, *i.e.* that for any open set $G \subset S_{N-1}$ we have

$$\big\{ r \in (R_0, R) : F(r) \cap G \neq \emptyset \big\} \text{ is measurable.} \tag{2.62}$$

For this purpose, consider a sequence of points $z_l \in S_{N-1}$, which is dense in S_{N-1}. Then, taking into account that

$$\sup_{z \in S_{N-1}} \int_{B(z,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y) = \sup_{l \in \mathbb{N}} \int_{B(z_l,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y),$$

we deduce that $T_{n,k}$ is measurable. Now, consider an open set $G \subset S_{N-1}$ and observe that by continuity, $F(r) \cap G$ is not empty if and only if

$$\forall\, m \in \mathbb{N},\ \exists z_l \in G, \quad T_{n,k}(r) - \int_{B(z_l,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y) < \frac{1}{m}.$$

Therefore, we get that

$$\big\{ r \in (R_0, R) : F(r) \cap G \neq \emptyset \big\} = \bigcap_{m \in \mathbb{N}} \bigcup_{z_l \in G} \left\{ r \in (R_0, R) : T_{n,k}(r) - \int_{B(z_l,h_k)\cap S_{N-1}} \Lambda_n(ry)\, ds(y) < \frac{1}{m} \right\}$$

is measurable.

Since F is valued on closed sets and measurable, we can apply the selection measurable theorem to derive a measurable function $g : (R_0, R) \mapsto S_{N-1}$ satisfying

$$\int_{B(g(r),h_k)\cap S_{N-1}} \mathit{\Lambda}_n(ry)\, ds(y) = \sup_{z\in S_{N-1}} \int_{B(z,h_k)\cap S_{N-1}} \mathit{\Lambda}_n(ry)\, ds(y), \quad \text{a.e. } r \in (R_0, R).$$

This implies that

$$B := \big\{ ry \in C(R_0, R) : y \in B\big(g(r), h_k\big) \cap S_{N-1} \big\} \tag{2.63}$$

is a measurable set of $C(R_0, R)$, which by (2.58), (2.59) and rotation invariance satisfies

$$|B| \le \phi(h_k) \int_{R_0}^{R} r^{N-1}\, dr = \phi(h_k)\, \frac{R^N - R_0^N}{N} < \delta_k. \tag{2.64}$$

By (2.57) and (2.60) this yields

$$\frac{\varepsilon^2}{2^{2k}} > \int_B \Lambda_n\, dx \ge \int_{R_0}^{R} T_{n,k}(r)\, r^{N-1}\, dr \ge \frac{\varepsilon}{2^k}\, R_0^{N-1}\, |E_{n,k}|,$$

hence the desired estimate (2.61).

Now, for $\lambda > 0$, define the set

$$U_n := \left\{ r \in U \setminus \bigcup_{k\in\mathbb{N}} E_{n,k} : \int_{S_{N-1}} \mathit{\Lambda}_n(ry)\, ds(y) < \lambda \right\}. \tag{2.65}$$

Then, (2.55) is satisfied by definition, while

$$\begin{aligned}
|U_n| \;\; &\ge \left| \left\{ r \in U : \int_{S_{N-1}} \mathit{\Lambda}_n(ry)\, ds(y) < \lambda \right\} \right| - \sum_{k\in\mathbb{N}} |E_{n,k}| \\
&\ge |U| - \left| \left\{ r \in U : \int_{S_{N-1}} \mathit{\Lambda}_n(ry)\, ds(y) \ge \lambda \right\} \right| - \frac{\varepsilon}{R_0^{N-1}} \\
&\ge |U| - \frac{1}{R_0^{N-1}} \left(\frac{1}{\lambda} \int_{\{|x|\in U\}} \mathit{\Lambda}_n\, dx + \varepsilon \right),
\end{aligned}$$

which gives (2.54).

Let us now prove that (2.56) holds. For this purpose, fix $k \in \mathbb{N}$ and, using Vitali's covering theorem, consider $y_1, \cdots, y_{n_k} \in S_{N-1}$ such that

$$S_{N-1} \subset \bigcup_{i=1}^{n_k} B(y_i, h_k), \quad B(y_i, h_k/5) \cap B(y_j, h_k/5) = \emptyset, \;\; \text{if } i \ne j.$$

Then, we have

$$
\begin{aligned}
&\|v_n - v\|^{q^*_{N-1}}_{L^\infty(U_n;L^{q^*_{N-1}}(S_{N-1}))} \le \left\| \sum_{i=1}^{n_k} \|v_n - v\|^{q^*_{N-1}}_{L^{q^*_{N-1}}(B(y_i,h_k)\cap S_{N-1}))} \right\|_{L^\infty(U_n)} \\
&\le 3^{q^*_{N-1}-1} \left\| \sum_{i=1}^{n_k} \left\| v_n - \frac{1}{\phi(h_k)} \int_{B(y_i,h_k)\cap S_{N-1}} v_n \, ds(y) \right\|^{q^*_{N-1}}_{L^{q^*_{N-1}}(B(y_i,h_k)\cap S_{N-1}))} \right\|_{L^\infty(U_n)} \\
&+ \frac{3^{q^*_{N-1}-1}}{\phi(h_k)^{q^*_{N-1}}} \sum_{i=1}^{n_k} \left\| \int_{B(y_i,h_k)\cap S_{N-1}} (v_n - v)\, ds(y) \right\|^{q^*_{N-1}}_{L^\infty(U_n)} \\
&+ 3^{q^*_{N-1}-1} \left\| \sum_{i=1}^{n_k} \left\| v - \frac{1}{\phi(h_k)} \int_{B(y_i,h_k)\cap S_{N-1}} v \, ds(y) \right\|^{q^*_{N-1}}_{L^{q^*_{N-1}}(B(y_i,h_k)\cap S_{N-1}))} \right\|_{L^\infty(U_n)}.
\end{aligned}
\tag{2.66}
$$

Using the invariance by dilatations of Sobolev-Wirtinger's inequality it follows from (2.65) and (2.60) that

$$
\begin{aligned}
&\left\| \sum_{i=1}^{n_k} \left\| v_n - \frac{1}{\phi(h_k)} \int_{B(y_i,h_k)\cap S_{N-1}} v_n \, ds(y) \right\|^{q^*_{N-1}}_{L^{q^*_{N-1}}(B(y_i,h_k)\cap S_{N-1}))} \right\|_{L^\infty(U_n)} \\
&\le \left\| \sum_{i=1}^{n_k} \|\nabla u_n(ry)\|^{q^*_{N-1}}_{L^q(B(y_i,h_k)\cap S_{N-1}))^N} \right\|_{L^\infty(U_n)} \\
&\le C \ \operatorname*{ess\text{-}sup}_{\substack{r\in U_n \\ z\in S_{N-1}}} \left(\|\nabla u_n(ry)\|^{q^*_{N-1}-q}_{L^q(B(z,h_k)\cap S_{N-1}))} \right) \|\nabla u_n\|^q_{L^\infty(U_n;L^q(S_{N-1}))} \\
&= C \ \operatorname*{ess\text{-}sup}_{\substack{r\in U_n \\ z\in S_{N-1}}} \left(T_{n,k}(r) \right)^{q^*_{N-1}-q} \|\nabla u_n\|^q_{L^\infty(U_n;L^q(S_{N-1}))} \le C \left(\frac{\varepsilon}{2^k} \right)^{q^*_{N-1}-q} \lambda.
\end{aligned}
$$

The same reasoning also shows that

$$
\left\| \sum_{i=1}^{n_k} \left\| v - \frac{1}{\phi(h_k)} \int_{B(y_i,h_k)\cap S_{N-1}} v \, ds(y) \right\|^{q^*_{N-1}}_{L^{q^*_{N-1}}(B(y_i,h_k)\cap S_{N-1}))} \right\|_{L^\infty(U_n)} \le C \left(\frac{\varepsilon}{2^k} \right)^{q^*_{N-1}-q} \lambda.
$$

Since v_n converges to v in $L^\infty(R_0, R_1; L^1(S_{N-1}))$, the second term in the right-hand side of (2.66) tends to zero. Therefore, taking the limsup as $\varepsilon \to 0$ in (2.66) we obtain that

$$
\limsup_{n\to\infty} \|v_n - v\|^{q^*_{N-1}}_{L^\infty(U_n;L^{q^*_{N-1}}(S_{N-1}))} \le C \left(\frac{\varepsilon}{2^k} \right)^{q^*_{N-1}-q} \lambda, \quad \forall\, k \in \mathbb{N},
$$

which finally yields (2.56). □

Lemma 2.14. *Let $N \ge 2$, and let $R_0, R > 0$ be such that $R_0 < R$. Consider a sequence u_n in $W^{1,N-1}(C(R_0,R))$ which converges weakly to a function u in $W^{1,N-1}(C(R_0,R))$ such that ∇u_n is bounded in $L^{N-1,1}(C(R_0,R))^N$ and satisfies the equi-integrability condition*

$$
\forall\, \varepsilon > 0,\ \exists\, \delta > 0, \quad \|\nabla u_n\|_{L^{N-1,1}(B)^M} \le \varepsilon,\ \ \forall\, n \in \mathbb{N},\ \forall\, B \subset C(R_0,R),\ |B| < \delta. \tag{2.67}
$$

Define $v_n, v \in L^{N-1}(R_0, R; W^{1,N-1}(S_{N-1}))$ by (2.24).
Then, for any closed set U of $[R_0, R]$ such that $v \in C^0(U; X^{1,N-1}(S_{N-1}))$, for any $\lambda, \varepsilon > 0$, there exists a sequence $U_n \subset U$ satisfying

$$
|U \setminus U_n| \le \frac{(R - R_0)^{\frac{N-2}{N-1}}}{R_0} \left(\frac{1}{\lambda} \|\nabla u_n\|_{L^{N-1,1}(\{|x|\in U\})^N} + \frac{1}{\lambda} \|\nabla u\|_{L^{N-1,1}(\{|x|\in U\})^N} + \varepsilon \right), \tag{2.68}
$$

$$\big\| \, |\nabla u_n(ry)| + |\nabla u(ry)| \, \big\|_{L^{N-1,1}(S_{N-1})} < \lambda, \quad a.e.\ r \in U_n, \tag{2.69}$$

$$\|v_n - v\|_{L^\infty(U_n; C^0(S_{N-1}))} \to 0. \tag{2.70}$$

Proof. The proof is quite similar to the one of Lemma 2.13. As before we first note that v_n converges to v in $C^0([R_0, R]; L^1(S_{N-1}))$.

Now, take $\varepsilon > 0$ and $\delta_k > 0$, $k \in \mathbb{N}$, such that for any measurable set $B \subset C(R_0, R)$ with $|B| < \delta_k$, we have

$$\|\Lambda_n\|_{L^{N-1,1}(B)} < \frac{\varepsilon^2}{2^{2k}}, \quad \forall\, n \in \mathbb{N}, \qquad \text{where } \Lambda_n := |\nabla u_n| + |\nabla u|. \tag{2.71}$$

Then, consider $h_k > 0$ such that (2.59) holds, and for $r \in (R_0, R)$, $n, k \in \mathbb{N}$, denote $T_{n,k}(r)$ by

$$T_{n,k}(r) := \operatorname*{ess-sup}_{z \in S_{N-1}} \|\Lambda_n(ry)\|_{L^{N-1,1}(B(z,h_k)\cap S_{N-1})}.$$

Now, the problem is to estimate the measure of the set $E_{n,k}$ defined by

$$E_{n,k} := \left\{ r \in (R_0, R) : T_{n,k}(r) > \frac{\varepsilon}{2^k} \right\}, \quad \text{for } k, n \in \mathbb{N}. \tag{2.72}$$

For this purpose, proceeding as in the proof of Lemma 2.13 we can construct a measurable function $g : (R_0, R) \to S_{N-1}$ such that for a.e. $r \in (0, R)$,

$$\|\Lambda_n(ry)\|_{L^{N-1,1}(B(g(r),h_k)\cap S_{N-1})} = \operatorname*{ess-sup}_{z \in S_{N-1}} \|\Lambda_n(ry)\|_{L^{N-1,1}(B(z,h_k)\cap S_{N-1})} = T_{n,k}(r).$$

The set B defined by (2.63) has a measure less than δ_k. Hence, using successively (2.71), Hölder's inequality and (2.72) if follows that

$$\frac{\varepsilon^2}{2^{2k}} \geq \|\Lambda_n\|_{L^{N-1,1}(B)} = \int_0^\infty \left(\int_{R_0}^R r^{N-1} \int_{B(g(r),h_k)\cap S_{N-1}} 1_{\{\Lambda_n > \lambda\}}\, ds(y)\, dr \right)^{\frac{1}{N-1}} d\lambda$$

$$\geq \frac{R_0}{(R-R_0)^{\frac{N-2}{N-1}}} \int_{R_0}^R \int_0^\infty \left(\int_{B(g(r),h_k)\cap S_{N-1}} 1_{\{\Lambda_n > \lambda\}}\, ds(y) \right)^{\frac{1}{N-1}} d\lambda\, dr$$

$$= \frac{R_0}{(R-R_0)^{\frac{N-2}{N-1}}} \int_{R_0}^R \|\Lambda_n\|_{L^{N-1,1}(B(g(r),h_k)\cap S_{N-1})}\, dr \geq \frac{R_0}{(R-R_0)^{\frac{N-2}{N-1}}} \frac{\varepsilon}{2^k} |E_{n,k}|,$$

which implies that

$$|E_{n,k}| < \frac{(R-R_0)^{\frac{N-2}{N-1}}}{R_0} \frac{\varepsilon}{2^k}, \quad \forall\, k, n \in \mathbb{N}. \tag{2.73}$$

A similar reasoning also shows that

$$\left| \left\{ r \in U : \|\Lambda_n(ry)\|_{L^{N-1,1}(S_{N-1})} \geq \lambda \right\} \right| \leq \frac{(R-R_0)^{\frac{N-2}{N-1}}}{\lambda R_0} \|\Lambda_n\|_{L^{N-1,1}(\{|x| \in U\})}, \quad \forall\, \lambda > 0. \tag{2.74}$$

Then, defining for $\lambda > 0$, the set

$$U_n := \left\{ r \in U \setminus \bigcup_{k \in \mathbb{N}} E_{n,k} : \|\Lambda_n(ry)\|_{L^{N-1,1}(S_{N-1})} < \lambda \right\},$$

we deduce from (2.73) and (2.74) that (2.68) and (2.69) hold. The proof of (2.70) is similar to the one of (2.56) taking into account that the space $X^{1,N-1}(S_{N-1})$ defined by (2.53) is continuously imbedded in $C^0(S_{N-1})$. $\square$

2.4 A counterexample

In the previous section we have needed some equi-integrability condition to extend the div-curl result of Theorem 2.1 to the case

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{N-1}.$$

Actually, the following counterexample shows that the conclusion of Theorem 2.1 is violated in general if the sequences σ_n and η_n are only bounded in $L^p(\Omega)^{M\times N}$ and $L^q(\Omega)^{M\times N}$ with (2.44).

Let $N \geq 2$ and $p, q \geq 1$ be such that (2.44) holds. Let $\Omega := B'_1 \times (0,1)$, where B'_1 the unit ball of $\mathbb{R}^{N-1}$ centered at the origin. The points of Ω are denoted by (x', x_N). We also denote by x' a point of Ω whose last coordinate is zero. Consider the functions σ_n and η_n, $n \geq 1$, defined in cylindrical coordinates by

$$\begin{cases} \sigma_n(x) := n^{\frac{N-1}{p}}\, a_n(|x'|)\, e_N \\ \eta_n(x) := n^{\frac{N-1}{p'}} \left(a'_n(|x'|)\, x_N\, \dfrac{x'}{|x'|} + a_n(|x'|)\, e_N \right), \end{cases} \quad \text{where} \quad a_n(r) := (1-r)^n. \tag{2.75}$$

Then, we have

Proposition 2.15. *The sequences σ_n and η_n defined by* (2.75) *satisfy*

$$\operatorname{div} \sigma_n = 0, \quad \operatorname{curl} \eta_n = 0 \quad \text{in } \Omega, \tag{2.76}$$

$$\begin{cases} \sigma_n \rightharpoonup 0 & \text{in } L^p(\Omega)^N, \quad \text{if } p > 1 \\ \sigma_n \overset{*}{\rightharpoonup} |S_{N-2}|\,(N-2)!\,\big(\delta_{\{x'=0\}} \otimes 1\big) & \text{in } \mathscr{M}(\Omega)^N, \quad \text{if } p = 1, \end{cases} \tag{2.77}$$

$$\begin{cases} \eta_n \rightharpoonup 0 \ \text{ in } L^q(\Omega)^N, & \text{if } q > 1 \\ \eta_n \overset{*}{\rightharpoonup} 0 \ \text{ in } \mathscr{M}(\Omega)^N, & \text{if } q = 1. \end{cases} \tag{2.78}$$

while

$$\sigma_n \cdot \eta_n \overset{*}{\rightharpoonup} |S_{N-2}|\, \frac{(N-2)!}{2^{N-1}} \big(\delta_{\{x'=0\}} \otimes 1\big) \ \text{ in } \mathscr{M}(\Omega)^N \quad (\text{with } |S_0| := 1). \tag{2.79}$$

Proof. It is clear that σ_n is divergence free and η_n is curl free in Ω. Moreover, a lengthy but easy computation shows that convergences (2.77), (2.78), (2.79) are a simple consequence of

$$(1-|x'|)^n \to 0, \quad \forall\, x' \in \mathbb{R}^{N-1}, \text{ with } 0 < |x'| < 1, \tag{2.80}$$

$$n^{k+1} \int_0^1 r^k (1-r)^{n\alpha}\, dr \to \frac{k!}{\alpha^{k+1}}, \quad \forall\, k \in \mathbb{N},\ \forall\, \alpha \geq 0. \tag{2.81}$$

□

3 Applications

3.1 Homogenization of systems with non equi-bounded coefficients

3.1.1 A Γ-convergence approach

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$. Consider a symmetric non-negative tensor-valued function $\mathbb{A}_n$ in $L^\infty(\Omega)^{(M\times N)^2}$ which satisfies the two following conditions:

- There exists a constant $\alpha > 0$ such that

$$\alpha \int_\Omega |Dv|^2\, dx \le \int_\Omega \mathbb{A}_n Dv : Dv\, dx, \quad \forall\, v \in H_0^1(\Omega)^M. \tag{3.1}$$

- There exists a non-negative Radon measure Λ on Ω satisfying

$$\begin{cases} |\mathbb{A}_n| \overset{*}{\rightharpoonup} \Lambda \quad \text{in } \mathscr{M}(\Omega), & \text{if } N = 2 \\ |\mathbb{A}_n|^\rho \overset{*}{\rightharpoonup} \Lambda \quad \text{in } \mathscr{M}(\Omega), \quad \text{with } \rho > \frac{N-1}{2}, & \text{if } N > 2. \end{cases} \tag{3.2}$$

Consider the quadratic functional F_n defined in $L^2(\Omega)^M$ by

$$F_n(v) := \begin{cases} \displaystyle\int_\Omega \mathbb{A}_n Dv : Dv\, dx, & \text{if } v \in H_0^1(\Omega)^M \\ \infty, & \text{if } v \in L^2(\Omega)^M \setminus H_0^1(\Omega)^M. \end{cases} \tag{3.3}$$

By a classical compactness result of Γ-convergence (see, *e.g.*, [24, 6]) there exist a subsequence of n, still denoted by n, and a quadratic functional $F : L^2(\Omega)^M \to [0, \infty]$ such that F_n Γ-converges to F for the strong topology of $L^2(\Omega)^M$, namely for any $v \in L^2(\Omega)^M$,

$$\begin{cases} \forall\, v_n \to v \quad \text{in } L^2(\Omega)^M, \quad F(v) \le \liminf\limits_{n\to\infty} F_n(v_n), \\ \exists\, \bar{v}_n \to v \quad \text{in } L^2(\Omega)^M, \quad F(v) = \lim\limits_{n\to\infty} F_n(\bar{v}_n). \end{cases} \tag{3.4}$$

Since F is quadratic, it has a bilinear form associated $\Psi : D(F) \times D(F) \to \mathbb{R}$. We recall that $D(F)$ is a Hilbert space endowed with the scalar product defined by Ψ.

Any sequence $\bar{v}_n$ satisfying the second statement of (3.4) is called a *recovery* sequence for F_n of limit v. Moreover, let $\bar{v}_n$ be a sequence in $L^2(\Omega)^M$ satisfying

$$\begin{cases} \bar{v}_n \to v \text{ strongly in } L^2(\Omega)^M \\ \sup\limits_{n\ge 0} F_n(\bar{v}_n) < \infty. \end{cases} \tag{3.5}$$

If $\bar{v}_n$ is a recovery sequence for F_n of limit v, then

$$\lim_{n\to\infty} \int_\Omega \mathbb{A}_n D\bar{v}_n : Dw_n\, dx = \Psi(v, w), \qquad \forall\, w_n \in L^2(\Omega)^M, \quad \begin{cases} w_n \to w \text{ in } L^2(\Omega)^M \\ \sup\limits_{n\ge 0} F_n(w_n) < \infty. \end{cases} \tag{3.6}$$

Reciprocally, if $\bar{v}_n$ satisfies

$$\lim_{n\to\infty} \int_\Omega \mathbb{A}_n D\bar{v}_n : Dw_n\, dx = 0, \qquad \forall\, w_n \in L^2(\Omega)^M, \quad \begin{cases} w_n \to 0 \text{ strongly in } L^2(\Omega)^M \\ \sup\limits_{n\ge 0} F_n(w_n) < \infty, \end{cases} \tag{3.7}$$

then $\bar{v}_n$ is a recovery sequence.

Define the number p by

$$p := \begin{cases} 1, & \text{if } N = 2 \\ \dfrac{2\rho}{1+\rho} \in \left(\dfrac{2N-2}{N+1}, 2\right), & \text{if } N > 2. \end{cases} \tag{3.8}$$

Then, we have the following compactness result:

Theorem 3.1. *Assume that conditions (3.1), (3.2) hold. Then there exists a subsequence of n, still denoted by n, such that*

$$F_n \xrightarrow{\Gamma} F, \tag{3.9}$$

and there exist a symmetric non-negative bilinear operator $\nu : D(F) \to \mathscr{M}(\Omega)$, a linear operator

$$\sigma \text{ which maps } D(F) \text{ into } \begin{cases} \mathscr{M}(\Omega)^{M\times N}, & \text{if } N=2,\ \Lambda \notin L^1(\Omega) \\ L^p(\Omega)^{M\times N}, & \text{if } N>2 \text{ or } N=2,\ \Lambda \in L^1(\Omega), \end{cases} \tag{3.10}$$

and a tensor-valued function

$$\mathbb{A} \in \begin{cases} \mathscr{M}(\Omega)^{(M\times N)^2}, & \text{if } N=2,\ \Lambda \notin L^1(\Omega) \\ L^1(\Omega)^{(M\times N)^2}, & \text{if } N=2,\ \Lambda \in L^1(\Omega) \\ L^\rho(\Omega)^{(M\times N)^2}, & \text{if } N>2, \end{cases} \tag{3.11}$$

satisfying the following conditions:

- *The operators ν and σ are strongly local in the sense*

$$\begin{cases} u, v \in D(F) \\ Du = Dv \text{ a.e. in } \omega \subset \Omega, \text{ open} \end{cases} \Rightarrow \begin{cases} \nu(u,u) = \nu(v,v) \\ \sigma(u) = \sigma(v) \end{cases} \text{ in } \omega. \tag{3.12}$$

- *The operator ν satisfies the ellipticity condition*

$$\alpha \int_\Omega |Dv|^2\,dx \le \int_\Omega d\nu(u,u), \quad \forall\, u \in D(F). \tag{3.13}$$

- *The tensor-valued measure $\mathbb{A}$ satisfy the following bounds*

$$\begin{cases} |\mathbb{A}| \le \Lambda & \text{in } \Omega, & \text{if } N=2 \\ |\mathbb{A}| \le (\Lambda^L)^{\frac{1}{\rho}} & \text{a.e. in } \Omega, & \text{if } N>2, \end{cases} \tag{3.14}$$

  where Λ^L is the absolute continuous part of Λ with respect to Lebesgue's measure.

- *The operators ν, σ and the tensor $\mathbb{A}$ are related by*
  - *For any $u, v \in D(F)$ and any open set $\omega \subset \Omega$,*

$$\left.\begin{array}{ll} v \in C^1(\omega)^M, & \text{if } N=2,\ \Lambda \notin L^1(\Omega) \\ v \in W^{1,p'}(\omega)^M, & \text{if } N>2 \text{ or } N=2,\ \Lambda \in L^1(\Omega) \end{array}\right\} \Rightarrow \nu(u,v) = \sigma(u) : Dv \text{ in } \omega. \tag{3.15}$$

  - *If $N=2$, $\Lambda \notin L^1(\Omega)$, we have for any open set $\omega \subset \Omega$,*

$$\sigma(u) = \mathbb{A}Du \text{ in } \omega, \quad \forall\, u \in D(F) \cap C^1(\omega)^M. \tag{3.16}$$

  - *If $N=2$, $\Lambda \in L^1(\Omega)$ or $N>2$, we have*

$$\sigma(u) = \mathbb{A}Du \text{ a.e. in } \Omega, \quad \forall\, u \in W_0^{1,p'}(\Omega)^M. \tag{3.17}$$

Moreover, denoting by ν^L the absolute continuous part of ν with respect to Lebesgue's measure, we have

$$\mathbb{A}Du : Dv \in L^1(\Omega), \quad \nu^L(u,v) = \mathbb{A}Du : Dv \;\; a.e. \text{ in } \Omega, \qquad \forall\, u,v \in D(F), \tag{3.18}$$

- *The functional F is given by*

$$F(u) = \int_\Omega d\nu(u,u), \quad \forall\, u \in D(F). \tag{3.19}$$

- *For any recovery sequence u_n for F_n of limit $u \in D(F)$, we have*

$$\mathbb{A}_n Du_n : Du_n \overset{*}{\rightharpoonup} \nu(u,u) \quad \text{in } \mathscr{M}(\Omega), \tag{3.20}$$

$$\mathbb{A}_n Du_n \rightharpoonup \sigma(u) \quad \begin{cases} \text{weakly} * \text{ in } \mathscr{M}(\Omega)^{M\times N}, & \text{if } N = 2 \\ \text{weakly in } L^{p'}(\Omega)^{M\times N}, & \text{if } N > 2. \end{cases} \tag{3.21}$$

Remark 3.2. Assuming $N > 2$ or $N = 2$, $\Lambda \in L^1(\Omega)$. We deduce from (3.15), (3.17) and (3.19), the following integral representation of F

$$F(u) = \int_\Omega \mathbb{A}Du : Du\, dx, \quad \forall\, u \in W_0^{1,p'}(\Omega)^M. \tag{3.22}$$

If $N = 2$, $\Lambda \notin L^1(\Omega)$, the above representation is also true for $u \in D(F) \cap C^1(\Omega)$, but in this case the integral must be understood as an integral with respect to the measure $\mathbb{A}$ and not with respect to Lebesgue's measure.

Remark 3.3. Let f_n be a sequence which converges strongly to some f in $H^{-1}(\Omega)^M$ and let u_n be the solution of

$$\begin{cases} -\operatorname{Div}(\mathbb{A}_n Du_n) = f_n \;\text{ in } \Omega \\ u_n \in H_0^1(\Omega)^M. \end{cases} \tag{3.23}$$

By (3.1) $F_n(u_n)$ is bounded, and thus, up to a subsequence, there exists $u \in D(F)$ such that u_n converges weakly to u in $H_0^1(\Omega)$. Since F_n Γ-converges to F, this implies that u_n is a recovery sequence for F_n and that u is the solution of

$$\begin{cases} u \in D(F) \\ \Psi(u,v) = \langle f, v\rangle, \quad \forall\, v \in D(F), \end{cases} \tag{3.24}$$

where Ψ the bilinear form associated with F. By a uniqueness argument it is not necessary to extract any subsequence. Moreover, convergence (3.21) implies that u is a solution of

$$-\operatorname{Div}\sigma(u) = f \;\text{ in } \Omega, \tag{3.25}$$

which taking into account (3.16), (3.17), can be read as

$$-\operatorname{Div}(\mathbb{A}Du) = f \;\text{ in } \Omega, \tag{3.26}$$

in the following cases: $N > 2$, $N = 2$ and $\Lambda \in L^1(\Omega)$, $N = 2$ and $u \in C^1(\Omega)$.

Remark 3.4. When $N = 2$, the boundedness of $\mathbb{A}_n$ in $L^1(\Omega)^{(M\times N)^2}$ ensures the convergence (3.21) of the flux. Similar compactness results in dimension two were obtained in the conductivity case [11, 12] and in the elasticity case [10]. When $N > 2$, convergence (3.21) holds when $\mathbb{A}_n$ is bounded in $L^\rho(\Omega)^{M\times N}$ with $\rho > (N-1)/2$. This condition is stronger than the equi-integrability of $\mathbb{A}_n$ in $L^1(\Omega)^{M\times N}$, which leads to a compactness result in the scalar case of [19] ($M = 1$). The proof of the scalar case is based on the maximum principle which does not hold for systems ($M > 1$).

Proof of Theorem 3.1. First all, note that by Hölder's inequality and (3.8) we have

$$\int_\Omega \mathbb{A}_n Du : Du\,dx \leq \begin{cases} \left(\displaystyle\int_\Omega |\mathbb{A}_n|\,dx\right)\|Du\|_{L^\infty(\Omega)^{M\times N}}, & \text{if } N = 2 \\ \left(\displaystyle\int_\Omega |\mathbb{A}_n|^\rho\,dx\right)^{\frac{1}{\rho}}\left(\displaystyle\int_\Omega |Du|^{p'}\,dx\right)^{\frac{2}{p'}}, & \text{if } N > 2, \end{cases} \qquad \forall\, u \in H^1_0(\Omega)^M,$$

which combined with condition (3.2) implies that the domain of the Γ-limit satisfies

$$D(F) \supset \begin{cases} W^{1,\infty}_0(\Omega)^M, & \text{if } N = 2 \\ W^{1,p'}_0(\Omega)^M, & \text{if } N > 2. \end{cases} \tag{3.27}$$

As above mentioned the existence of a subsequence of n and a functional F satisfying (3.9) is well known. The proof is divided in three steps.

First step: Determination of the operators σ and μ.
From (3.1) we deduce the inequality

$$\alpha\int_\Omega |Dv|^2\,dx \leq F(v), \quad \forall\, v \in D(F), \tag{3.28}$$

which combined with $C^1_0(\Omega)^M \subset D(F)$ shows that $D(F)$ is continuously and densely imbedded in $H^1_0(\Omega)^M$, and thus that $H^{-1}(\Omega)^M$ is continuously and densely imbedded in $D(F)'$.

Denoting by $\Psi : D(F) \times D(F) \to \mathbb{R}$ the bilinear form associated with F and taking a countable dense subset $\mathscr{E}$ of $L^2(\Omega)^M$, define the set E by

$$E := \left\{ u \in D(F) : \exists\, f \in \mathscr{E},\ \ \Psi(u,v) = \int_\Omega f\cdot v\,dx,\ \forall\, v \in D(F) \right\} \tag{3.29}$$

which is a dense and countable subset of $D(F)$.

For $f \in \mathscr{E}$, consider the solution u_n of

$$\begin{cases} -\operatorname{Div}(\mathbb{A}_n Du_n) = f \ \text{ in } \Omega \\ u_n \in H^1_0(\Omega)^M. \end{cases} \tag{3.30}$$

By (3.1) the sequence u_n satisfies the estimate

$$\int_\Omega \mathbb{A}_n Du_n : Du_n\,dx + \int_\Omega |Du_n|^2\,dx \leq C. \tag{3.31}$$

Hence, up to a subsequence, there exist $u \in H^1_0(\Omega)^M$ and $\mu_u \in \mathscr{M}(\Omega)^M$ such that

$$u_n \rightharpoonup u \ \text{ in } H^1_0(\Omega)^M, \tag{3.32}$$

$$\mathbb{A}_n Du_n : Du_n \overset{*}{\rightharpoonup} \mu_u \ \text{ in } \mathscr{M}(\Omega). \tag{3.33}$$

Taking into account that (3.30) implies (3.7), we deduce that u_n is a recovery sequence for F_n of limit u and that

$$\Psi(u,v) = \int_\Omega f \cdot v\, dx, \quad \forall\, v \in D(F), \tag{3.34}$$

Hence, u is the element of E associated with the function $f \in \mathscr{E}$ and

$$\mu_u(\Omega) = F(u). \tag{3.35}$$

By Hölder's inequality we have for any $\phi \in C_0^0(\Omega)$, $\phi \geq 0$,

$$\begin{aligned}
&\int_\Omega |\mathbb{A}_n Du_n|^p\, \phi\, dx \leq \int_\Omega (\mathbb{A}_n Du_n : Du_n)^{\frac{p}{2}}\, |\mathbb{A}_n|^{\frac{p}{2}}\, \phi\, dx \\
&\leq \left(\int_\Omega \mathbb{A}_n Du_n : Du_n\, \phi\, dx \right)^{\frac{p}{2}} \left(\int_\Omega |\mathbb{A}_n|^{\frac{p}{2-p}}\, \phi\, dx \right)^{1-\frac{p}{2}} \\
&= \left(\int_\Omega \mathbb{A}_n Du_n : Du_n\, \phi\, dx \right)^{\frac{p}{2}} \left(\int_\Omega |\mathbb{A}_n|^{\rho}\, \phi\, dx \right)^{1-\frac{p}{2}}.
\end{aligned}$$

Hence, we deduce the existence of σ_u such that

$$\begin{cases} \mathbb{A}_n Du_n \overset{*}{\rightharpoonup} \sigma_u \ \text{ in } \mathscr{M}(\Omega)^{M\times N}, & \text{if } N = 2 \\ \mathbb{A}_n Du_n \rightharpoonup \sigma_u \ \text{ in } L^p(\Omega)^{M\times N},\ |\mathbb{A}_n Du_n|^p \text{ equi-integrable}, & \text{if } N > 2, \end{cases} \tag{3.36}$$

and by (3.2) for any $\Phi \in C_0^0(\Omega)^{M\times N}$,

$$\int_\Omega \sigma_u : \Phi\, dx \leq \begin{cases} \left(\int_\Omega |\Phi|\, d\mu_u \right)^{\frac{1}{2}} \left(\int_\Omega |\Phi|\, d\Lambda \right)^{\frac{1}{2}}, & \text{if } N = 2 \\ \left(\int_\Omega |\Phi|\, d\mu_u \right)^{\frac{1}{2}} \left(\int_\Omega |\Phi|\, d\Lambda \right)^{\frac{1}{p}-\frac{1}{2}} \left(\int_\Omega |\Phi|\, dx \right)^{\frac{1}{p'}} & \text{if } N > 2. \end{cases} \tag{3.37}$$

By (3.27) and (3.30) σ_u also satisfies

$$\int_\Omega \sigma_u : Dv\, dx = \Psi(u,v), \quad \forall\, v \in \begin{cases} C_0^1(\Omega)^M, & \text{if } N = 2 \\ W_0^{1,p'}(\Omega)^M, & \text{if } N > 2. \end{cases} \tag{3.38}$$

Since $\mathscr{E}$ is countable, these subsequences can be chosen independently of f. Moreover, taking two elements $f, g \in \mathscr{E}$, and denoting by u, μ_u, σ_u and by v, μ_v, σ_v the above defined elements associated with f and g respectively, we have

$$\begin{aligned}
&\|\mu_u - \mu_v\|_{\mathscr{M}(\Omega)} \leq \liminf_{n\to\infty} \int_\Omega \left| A_n Du_n : Du_n - A_n Dv_n : Dv_n \right| dx \\
&= \liminf_{n\to\infty} \int_\Omega \left| A_n D(u_n + v_n) : D(u_n - v_n) \right| dx \\
&\leq \lim_{n\to\infty} \left(\int_\Omega A_n D(u_n + v_n) : D(u_n + v_n)\, dx \right)^{\frac{1}{2}} \left(\int_\Omega A_n D(u_n - v_n) : D(u_n - v_n)\, dx \right)^{\frac{1}{2}} \\
&= \|u + v\|_{D(F)}\, \|u - v\|_{D(F)}.
\end{aligned} \tag{3.39}$$

and (in the case $N=2$, $L^p(\Omega)^{M\times N}$ must be replaced by $\mathscr{M}(\Omega)^{M\times N}$)

$$\begin{aligned}\|\sigma_u-\sigma_v\|^p_{L^p(\Omega)^{M\times N}} &\le \liminf_{n\to\infty}\int_\Omega \big|\mathbb{A}_n D(u_n-v_n)\big|^p\,dx\\ &\le \liminf_{n\to\infty}\left(\int_\Omega \mathbb{A}_n D(u_n-v_n):D(u_n-v_n)\,dx\right)^{\frac{p}{2}}\left(\int_\Omega |\mathbb{A}_n|^\rho\,dx\right)^{1-\frac{p}{2}}\\ &\le C\,\|u-v\|^p_{D(F)}.\end{aligned}\tag{3.40}$$

This estimate allows us to extend by continuity the operators

$$u\in E\mapsto \mu_u\in\mathscr{M}(\Omega)\quad\text{and}\quad u\in E\mapsto\sigma_u\in\begin{cases}\mathscr{M}(\Omega)^{M\times N}, & \text{if } N=2\\ L^p(\Omega)^{M\times N}, & \text{if } N>2,\end{cases}$$

to operators defined in the whole domain $D(F)$, that we denote by σ and μ. It is easy to check that μ is quadratic and σ is linear. Moreover, by (3.39) and (3.40) μ and σ satisfy

$$\|\mu(u)-\mu(v)\|_{\mathscr{M}(\Omega)}\le\|u+v\|_{D(F)}\,\|u-v\|_{D(F)},\quad\forall\,u,v\in D(F),\tag{3.41}$$

$$\begin{cases}\|\sigma(u)\|_{\mathscr{M}(\Omega)^{M\times N}}\le C\,\|u\|_{D(F)}, & \text{if } N=2\\ \|\sigma(u)\|_{L^p(\Omega)^{M\times N}}\le C\,\|u\|_{D(F)}, & \text{if } N>2,\end{cases}\tag{3.42}$$

$$\int_\Omega d\mu(u)=F(u),\quad\forall\,u\in D(F),\tag{3.43}$$

$$\int_\Omega\sigma(u):Dv\,dx=\Psi(u,v),\quad\forall\,v\in\begin{cases}C^1_0(\Omega)^M & \text{if } N=2\\ W^{1,p}_0(\Omega)^M, & \text{if } N>2,\end{cases}\quad\forall\,u\in D(F).\tag{3.44}$$

Moreover, observe that for a given recovery sequence $\bar u_n$ of limit $\bar u\in D(F)$, taking $f\in\mathscr{E}$ and u_n, u the solutions of (3.30), (3.34), we have for any $\phi\in C^0_0(\Omega)$,

$$\begin{aligned}&\left|\int_\Omega\mathbb{A}_nD\bar u_n:D\bar u_n\,\phi\,dx-\int_\Omega\phi\,d\mu(\bar u)\right|\le\left|\int_\Omega\mathbb{A}_nD\bar u_n:D\bar u_n\,\phi\,dx-\int_\Omega\mathbb{A}_nDu_n:Du_n\,\phi\,dx\right|\\ &+\left|\int_\Omega\mathbb{A}_nDu_n:Du_n\,\phi\,dx-\int_\Omega\phi\,d\mu(u)\right|+\left|\int_\Omega\phi\,d\mu(u)-\int_\Omega\phi\,d\mu(\bar u)\right|.\end{aligned}\tag{3.45}$$

Using that $\bar u_n+u_n$ and $\bar u_n-u_n$ are recovery sequences of limits $\bar u+u$ and $\bar u-u$ respectively, we also have

$$\begin{aligned}&\left|\int_\Omega\mathbb{A}_nD\bar u_n:D\bar u_n\,\phi\,dx-\int_\Omega\mathbb{A}_nDu_n:Du_n\,\phi\,dx\right|\\ &\le\|\phi\|_{C^0_0(\Omega)}\int_\Omega\big|\mathbb{A}_nD(\bar u_n+u_n):D(\bar u_n-u_n)\big|\,dx\\ &\le\|\phi\|_{C^0_0(\Omega)}\left(\int_\Omega\mathbb{A}_nD(\bar u_n+u_n):D(\bar u_n+u_n)\,dx\right)^{\frac12}\left(\int_\Omega\mathbb{A}_n(D\bar u_n-u_n):D(\bar u_n-u_n)\,dx\right)^{\frac12}\\ &\to\|\phi\|_{C^0_0(\Omega)}\,\|\bar u+u\|_{D(F)}\,\|\bar u-u\|_{D(F)}.\end{aligned}$$

Therefore, taking the limsup in (3.45) and using (3.33), (3.41), we get that

$$\limsup_{n\to\infty}\left|\int_\Omega\mathbb{A}_nD\bar u_n:D\bar u_n\,\phi\,dx-\int_\Omega\phi\,d\mu(\bar u)\right|\le 2\,\|\phi\|_{C^0_0(\Omega)}\,\|\bar u+u\|_{D(F)}\,\|\bar u-u\|_{D(F)},$$

which by the density of E in $D(F)$ implies that

$$\lim_{n\to\infty}\int_\Omega \mathbb{A}_n D\bar{u}_n : D\bar{u}_n\,\phi\,dx = \int_\Omega \phi\,d\mu(\bar{u}), \quad \forall\,\phi\in C_0^0(\Omega),$$

Therefore, (3.33) holds for any $u\in D(F)$ and any recovery sequence u_n of limit u. Analogously, (3.36) holds for any $u\in D(F)$ and any recovery sequence u_n of limit u.

From the quadratic mapping $u\in D(F)\mapsto\mu(u)\in\mathscr{M}(\Omega)$, we can now construct the associated bilinear operator $\nu: D(F)\times D(F)\to\mathscr{M}(\Omega)$, defined by

$$\nu(u,v) := \frac{1}{4}\big(\mu(u+v)-\mu(u-v)\big), \quad \forall\,u,v\in D(F), \tag{3.46}$$

which satisfies

$$\mathbb{A}_n Du_n : Dv_n \overset{*}{\rightharpoonup} \nu(u,v) \quad \text{in } \mathscr{M}(\Omega), \tag{3.47}$$

for any $u,v\in D(F)$ and any recovery sequences u_n and v_n of limits u and v respectively.

Second step: Use of the div-curl result for the derivation of σ and μ.
Let u_n be the solution of equation (3.30) with $f\in E\subset D(F)$, and let v_n be a recovery sequence of limit $v\in D(F)$. Let us check that the sequences $\sigma_n := \mathbb{A}_n Du_n$ and $\eta_n := Dv_n$ satisfy the assumptions of Theorem 2.1:

First, by convergences (3.32) and (3.36) σ_n and η_n satisfy (2.4) and (2.5), where $p\in(1,2)$ and $q=2$ are such that

$$\frac{1}{p}+\frac{1}{2} = 1+\frac{1}{2\rho} < 1+\frac{1}{N-1},$$

as well as condition (2.2) with $s_n=2$. Next, by the Cauchy-Schwarz inequality combined with the boundedness of $F_n(u_n)$ and $F_n(v_n)$, the sequence $\sigma_n:\eta_n = \mathbb{A}_n Du_n : Dv_n$ is bounded in $L^1(\Omega)$, so that convergence (2.3) holds (see the comment after (1.15)).

Then, the limit formulation (2.8) of Theorem 2.1 shows that

$$\int_\Omega \psi\,d\nu(u,v) = \int_\Omega f\cdot v\,\psi\,dx - \int_\Omega \big(\sigma(u)\nabla\psi\big)\cdot v\,dx, \tag{3.48}$$

where $\psi(x):=\varphi(|x-x_0|)$, and φ satisfying the conditions (2.8) or (2.14) depending if $N>2$ or $N=2$.

In particular, we can take in (3.48) a function $v\in D(F)$ such that for some open set $\omega\subset\Omega$,

$$v\in\begin{cases} C^1(\omega)^M, & \text{if } N=2,\ \Lambda\notin L^1(\Omega) \\ W^{1,p'}(\omega)^M, & \text{if } N>2 \text{ or } N=2,\ \Lambda\in L^1(\Omega), \end{cases} \tag{3.49}$$

a ball $B(x_0,R)\subset\omega$, and a function $\varphi\in W^{1,\infty}(0,\infty)$ with $\operatorname{supp}\varphi\subset[0,R]$. Also using that (3.34) and (3.38) imply

$$-\operatorname{Div}\sigma(u) = f \quad \text{in } \mathscr{D}'(\Omega), \tag{3.50}$$

and that by (3.37) (which by continuity holds for any $u\in D(F)$) we have $\sigma(u)\in L^1(\Omega)^{M\times N}$ if $\Lambda\in L^1(\Omega)$, we get that

$$\int_\Omega \psi\,d\nu(u,v) = \int_\Omega \sigma(u):D(v\psi)\,dx - \int_\Omega \big(\sigma(u)\nabla\psi\big)\cdot v\,dx = \int_\Omega \sigma(u):Dv\,\psi\,dx.$$

Taking in this equality $u=v_k\in E$ converging to v in $D(F)$, it follows that

$$\int_\Omega \psi\,d\mu(v) = \int_\Omega \sigma(v):Dv\,\psi\,dx, \tag{3.51}$$

for any radial function ψ with respect to some $x_0 \in \omega$ and with support in ω. By (3.37) we then have

$$\int_\Omega \sigma(v) : \Phi \, dx \leq \begin{cases} \left(\displaystyle\int_\Omega \sigma(v) : Dv \, |\Phi| \, dx \right)^{\frac{1}{2}} \left(\displaystyle\int_\Omega |\Phi| \, d\Lambda \right)^{\frac{1}{2}}, & \text{if } N = 2 \\ \left(\displaystyle\int_\Omega \sigma(v) : Dv \, |\Phi| \, dx \right)^{\frac{1}{2}} \left(\displaystyle\int_\Omega |\Phi| \, d\Lambda \right)^{\frac{1}{p} - \frac{1}{2}} \left(\displaystyle\int_\Omega |\Phi| \, dx \right)^{\frac{1}{p'}}, & \text{if } N > 2, \end{cases}$$

for any $\Phi \in C_0^0(\omega)^{M \times N}$ radial with respect to some $x_0 \in \omega$. By the measures derivation theorem this yields

$$\begin{cases} \sigma(v) = H_v \Lambda \ \text{ with } \ |H_v| \leq \big(H_v : Dv \big)^{\frac{1}{2}} & \Lambda\text{-a.e. in } \omega, \quad \text{if } N = 2 \\ |\sigma(v)| \leq \big(\sigma(v) : Dv \big)^{\frac{1}{2}} \, (\Lambda^L)^{\frac{1}{p} - \frac{1}{2}} & \text{a.e. in } \omega, \quad \text{if } N > 2, \end{cases} \tag{3.52}$$

Therefore, for any function v satisfying (3.49), we obtain the estimates

$$\begin{cases} \sigma(v) = H_v \Lambda \ \text{ with } \ |H_v| \leq |Dv| & \Lambda\text{-a.e. in } \omega, \quad \text{if } N = 2 \\ |\sigma(v)| \leq |Dv| \, (\Lambda^L)^{\frac{2}{p} - 1} & \text{a.e. in } \omega, \quad \text{if } N > 2. \end{cases} \tag{3.53}$$

Third step: Expressions of σ and μ in terms of the limit tensor $\mathbb{A}$.
Let Ω_k, $k \geq 1$, be an exhaustive sequence of open sets in Ω such that

$$\forall\, k \geq 1, \ \ \overline{\Omega_k} \subset \Omega_{k+1} \quad \text{and} \quad \bigcup_{k \geq 1} \Omega_k = \Omega. \tag{3.54}$$

We associate with the open sets Ω_k the functions ϕ_k satisfying

$$\forall\, k \geq 1, \quad \phi_k \in C_c^1(\Omega_{k+1}) \ \text{ and } \ \phi_k \equiv 1 \ \text{ in } \Omega_k. \tag{3.55}$$

Then, define the tensor-valued measure $\mathbb{A}$ by

$$\mathbb{A}(f_i \otimes e_j) := \sum_{k=1}^{\infty} \sigma_{(\phi_k x_j f_i)} 1_{\Omega_k}, \quad \text{for } 1 \leq i \leq M, \ 1 \leq j \leq N. \tag{3.56}$$

Given $\xi \in \mathbb{R}^{M \times N}$ and applying (3.53) to the functions $v - \phi_k \, \xi x$, we have

$$\begin{cases} \sigma(v) - \mathbb{A}\xi = H_{(v - \phi_k \xi x)} \Lambda \ \text{ with } \ |H_{v - \phi_k \xi x}| \leq |Dv - \xi| & \Lambda\text{-a.e. in } \omega \cap \Omega_k, \quad \text{if } N = 2 \\ |\sigma(v) - \mathbb{A}\xi| \leq |Dv - \xi| \, (\Lambda^L)^{\frac{2}{p} - 1} & \text{a.e. in } \omega \cap \Omega_k, \quad \text{if } N > 2. \end{cases}$$

Therefore, we get that

$$\sigma(v) = \mathbb{A} Dv \ \text{ in } \omega, \quad \text{for any } v \in D(F) \text{ satisfying (3.49)}. \tag{3.57}$$

By (3.51) we also have

$$\mu(v) = \mathbb{A} Dv : Dv \ \text{ in } \omega, \quad \text{for any } v \in D(F) \text{ satisfying (3.49)}. \tag{3.58}$$

As a consequence, we obtain that if $u_1, u_2 \in D(F)$ are such that there exists an open set $\omega \subset \Omega$ with $Du_1 = Du_2$ in ω, then $\sigma(u_1 - u_2) = 0$ and $\mu(u_1 - u_2) = 0$ on ω. Hence, from the Cauchy-Schwarz inequality we deduce that

$$\sigma(u_1) - \sigma(u_2) = \sigma(u_1 - u_2) = 0 \quad \text{in } \omega, \tag{3.59}$$

$$\begin{aligned} \|\mu(u_1)-\mu(u_2)\|_{\mathscr{M}(\omega)} &= \|\nu(u_1+u_2,u_1-u_2)\|_{\mathscr{M}(\omega)} \\ &\leq \|\mu(u_1+u_2)\|^{\frac{1}{2}}_{\mathscr{M}(\omega)}\,\|\mu(u_1-u_2)\|^{\frac{1}{2}}_{\mathscr{M}(\omega)} = 0, \end{aligned} \tag{3.60}$$

which establishes the local property (3.12) of σ and ν.

From now on, assume that $N>2$ or $N=2$ and $\Lambda\in L^1(\Omega)$, which implies that $\sigma(u)$ belongs to $L^p(\Omega)^{M\times N}$ for any $u\in D(F)$. For a function $u\in E$ and a ball $B(x_0,2R)\subset\Omega$, define

$$\bar{u} := \frac{1}{|B_{2R}|}\int_{B(x_0,2R)} u\,dx,$$

$$U := \Big\{ r\in(R,2R) : \int_{\partial B(x_0,r)} \big(|u-\bar{u}|^2 + r^2|Du|^2\big)\,ds(x) \\ \leq \frac{2}{R}\int_{B(x_0,2R)}\big(|u-\bar{u}|^2+|x-x_0|^2|Du|^2\big)\,dx \Big\}.$$

The set U satisfies

$$\big|(R,2R)\backslash U\big| \leq \frac{R/2}{\displaystyle\int_{B(x_0,2R)}\big(|u-\bar{u}|^2+|x-x_0|^2|Du|^2\big)\,dx}\int_R^{2R}\int_{\partial B(x_0,r)}\big(|u-\bar{u}|^2+r^2|Du|^2\big)\,ds(x)\,dr \leq \frac{R}{2},$$

hence $|U|\geq R/2$.

Next, define

$$\varphi(r) := \frac{1}{|U|}\int_r^{2R} 1_U\,ds,\quad \text{for } r\geq 0, \quad \text{and} \quad \psi(x) := \varphi(|x-x_0|),\quad \text{for } x\in\Omega,$$

and $v := u-\bar{u}\,\phi$, for $\phi\in C^1_c(\Omega)$ with $\phi\equiv 1$ in $B(x_0,2R)$. By the local property (3.12) we have $\mu(u)=\nu(u,v)$ in $B(x_0,2R)$. Putting this in formula (3.48) and noting that $\psi\equiv 1$ in $B(x_0,R)$, we obtain

$$\big|\mu(u)\big(\bar{B}(x_0,R)\big)\big| \leq \int_\Omega \psi\,d\mu(u) = \int_\Omega f\cdot(u-\bar{u})\,\psi\,dx + \int_\Omega \big(\sigma(u)\nabla\psi\big)\cdot(u-\bar{u})\,dx.$$

First, using Poincaré-Wirtinger's inequality in $B(x_0,2R)$ and Hölder's inequality in $\partial B(x_0,r)$, second Sobolev's imbedding of $H^1(\partial B(x_0,r))$ into $L^{p'}\big(\partial B(x_0,r)\big)$ (recall that $\frac{1}{p}>\frac{1}{2}+\frac{1}{N-1}$), third the definition of U, Hölder's inequality in $(R,2R)$ and again Poincaré-Wirtinger's inequality in

$B(x_0, 2R)$, we get that

$$\begin{aligned}
&\left|\mu(u)\big(\bar{B}(x_0,R)\big)\right| \\
&\le CR\left(\int_{B(x_0,2R)} |f|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B(x_0,2R)} |Du|^2\,dx\right)^{\frac{1}{2}} \\
&\quad + \frac{2}{R}\int_U \left(\int_{\partial B(x_0,r)} |\sigma(u)|^p\,ds(x)\right)^{\frac{1}{p}}\left(\int_{\partial B(x_0,r)} |u-\bar{u}|^{p'}\,ds(x)\right)^{\frac{1}{p'}}\,dr \\
&\le CR\left(\int_{B(x_0,2R)} |f|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B(x_0,2R)} |Du|^2\,dx\right)^{\frac{1}{2}} \\
&\quad + CR^{(N-1)(\frac{1}{p'}-\frac{1}{2})-1}\int_U \left(\int_{\partial B(x_0,r)} |\sigma(u)|^p\,ds(x)\right)^{\frac{1}{p}}\left(\int_{\partial B(x_0,r)} \left(|u-\bar{u}|^2 + r^2|Du|^2\right)ds(x)\right)^{\frac{1}{2}}\,dr \\
&\le CR\left(\int_{B(x_0,2R)} |f|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B(x_0,2R)} |Du|^2\,dx\right)^{\frac{1}{2}} \\
&\quad + CR^{N(\frac{1}{p'}-\frac{1}{2})}\left(\int_{B(x_0,2R)} |\sigma(u)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B(x_0,2R)} |Du|^2\,dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Dividing this inequality by $|B(x_0,R)|$ and passing to the limit as R tends to zero, we deduce that

$$\left|\mu^L(u)\right| \le C\,|\sigma(u)|\,|Du| \quad \text{a.e. in } \Omega, \tag{3.61}$$

where $\mu^L(u)$ denotes the absolute continuous component of $\mu(u)$ with respect to Lebesgue's measure. On the other hand, also remark that (3.37) also implies that

$$|\sigma(u)| \le \left|\mu^L(u)\right|^{\frac{1}{2}} (\Lambda^L)^{\frac{1}{p}-\frac{1}{2}} \quad \text{a.e. in } \Omega,$$

which combined to (3.61) gives

$$|\sigma(u)| \le C\,|Du|\,(\Lambda^L)^{\frac{2}{p}-1} \quad \text{a.e. in } \Omega.$$

This inequality is similar to (3.53) (which was proved for v smooth), and thus reasoning as for the derivation of (3.57), we get that

$$\sigma(u) = \mathbb{A}Du \;\text{ a.e. in } \Omega, \tag{3.62}$$

for any $u \in E$, and then by continuity for any $u \in D(F)$. Returning to (3.61) and taking into account the density of E in $D(F)$, we also have

$$\left|\mu^L(u)\right| \le C\,|\mathbb{A}Du|\,|Du| \;\text{ a.e. in } \Omega, \quad \forall\, u \in D(F).$$

Using this inequality with u replaced by $u - \xi x$, $\xi \in \mathbb{R}^{M\times N}$, and recalling that the mapping $u \mapsto \mu_L$ is quadratic and nonnegative, we obtain

$$\begin{aligned}
&\left|\mu^L(u) - \mathbb{A}\xi : \xi\right| \le \left|\mu^L(u) - \mu^L(\xi x)\right| \le \left|\mu^L(u+\xi x)\right|^{\frac{1}{2}}\left|\mu^L(u-\xi x)\right|^{\frac{1}{2}} \\
&\le C\left|\mathbb{A}(Du+\xi)\right|^{\frac{1}{2}}|Du+\xi|^{\frac{1}{2}}\left|\mathbb{A}(Du-\xi)\right|^{\frac{1}{2}}|Du-\xi|^{\frac{1}{2}} \quad \text{a.e. in } \Omega,\ \forall\, \xi \in \mathbb{R}^N.
\end{aligned}$$

This finally shows that

$$\mu^L(u) = \mathbb{A}Du : Du \;\text{ a.e. in } \Omega, \quad \forall\, u \in D(F). \tag{3.63}$$

3.1.2 A H-convergence approach

We have the following H-convergence type result. Is is similar to Theorem 3.1 but with different assumptions, which allow to treat the case of non-symmetric tensor-valued functions $\mathbb{A}_n$.

Theorem 3.5. *Let $\mathbb{A}_n$ be a non-negative tensor-valued function in $L^\infty(\Omega)^{(M\times N)^2}$ which satisfies conditions* (3.1) *and* (3.2) *with $\Lambda \in L^\infty(\Omega)$. Also assume that there exists a constant $C > 0$ such that*

$$(\mathbb{A}_n\xi : \eta)^2 \leq C\,(\mathbb{A}_n\xi : \xi)\,(\mathbb{A}_n\eta : \eta) \quad \text{a.e. in } \Omega,\ \forall\,\xi, \eta \in \mathbb{R}^{M\times N}. \tag{3.64}$$

Then, there exist a subsequence of n, still denoted by n, and a tensor-valued function $\mathbb{A}$ in $L^\infty(\Omega)^{(M\times N)^2}$ satisfying (3.1) *and*

$$|\mathbb{A}| \leq C\,\|\Lambda\|_{L^\infty(\Omega)} \quad \text{a.e. in } \Omega, \tag{3.65}$$

such that for any $f \in H^{-1}(\Omega)^M$, the solution u_n in $H^1_0(\Omega)^M$ of the equation

$$-\operatorname{Div}(\mathbb{A}_n Du_n) = f \quad \text{in } \Omega \tag{3.66}$$

satisfies the convergences

$$u_n \rightharpoonup u \ \text{ weakly in } H^1_0(\Omega)^M, \quad \mathbb{A}_n Du_n \rightharpoonup \mathbb{A}Du \ \begin{cases} \text{in } \mathscr{M}(\Omega)^{M\times N}\,*, & \text{if } N = 2 \\ \text{in } L^p(\Omega)^{M\times N}, & \text{if } N > 2, \end{cases} \tag{3.67}$$

where u is the solution in $H^1_0(\Omega)^M$ of the equation

$$-\operatorname{Div}(\mathbb{A}Du) = f \quad \text{in } \Omega. \tag{3.68}$$

Remark 3.6. The extra condition (3.64) compensates the fact that $\mathbb{A}_n$ is not necessarily symmetric. The price to pay with respect to the Γ-convergence result of Theorem 3.1 is that the limit Λ of (3.2) needs to be in $L^\infty(\Omega)$.

Remark 3.7. Theorem 3.5 is an extension to non equi-bounded coefficients of the classical H-convergence of Murat-Tartar [36, 45]. In dimension two Theorem 3.5 includes the scalar case of [11] and the elasticity case of [10]. In higher dimension it generalizes the H-convergence of [15] thanks to the improvement of the div-curl result.

The proof of Theorem 3.5 follows the same scheme as the Murat-Tartar H-convergence [36] and some of its extensions [11, 15, 10]. In particular it is quite similar to the proof of Theorem 5.2 [15] restricted to the linear case, using the new div-curl result of Theorem 2.1. So we omit it.

3.2 Weak continuity of the Jacobian

Let Ω be a regular bounded open set of $\mathbb{R}^N$, $N \geq 2$. It is well known that the distributional determinant defined for $u = (u^1, \cdots, u^N) : \Omega \to \mathbb{R}^N$ by (where cof denotes the cofactors matrix)

$$\operatorname{Det}(Du) := \sum_{j=1}^{N} \partial_j \left[u^i \operatorname{cof}(Du)_{ij}\right], \quad \text{for } 1 \leq i \leq N, \tag{3.69}$$

agrees with the determinant $\det(Du)$ if $u \in W^{1,N}(\Omega)^N$ (see, *e.g.*, [22], Lemma 2.7 for further details), but the situation is more delicate if u is less regular. There has been a lot of works about the distributional determinant, its link with the determinant and its weak continuity; we refer to [1, 2, 22, 23, 33, 35] for various contributions in the topic. In particular, Müller showed [35] that

$$\mathrm{Det}\,(Du) = \det\,(Du), \quad \forall\, u \in W^{1,s}(\Omega)^N,\ \forall\, s \geq \frac{N^2}{N+1}, \tag{3.70}$$

whenever $\mathrm{Det}\,(Du) \in L^1(\Omega)$. In connection with this result, one has (see [1, 23], and also [30, 28] for refinements)

$$u_n \rightharpoonup u \ \text{ in } W^{1,s}(\Omega)^N \quad \Rightarrow \quad \mathrm{Det}\,(Du_n) \rightharpoonup \mathrm{Det}\,(Du) \ \text{ in } \mathscr{D}'(\Omega), \quad \forall\, s > \frac{N^2}{N+1}. \tag{3.71}$$

Up to our knowledge the most recent result is due to Brezis and Nguyen [8] who proved that for any $s \in [N-1, \infty]$ and for any vector-valued functions u_n, u in $L^\infty(\Omega)^N$,

$$\left.\begin{array}{ll} u_n \rightharpoonup u & \text{in } W^{1,s}(\Omega)^N \\ u_n \to u & \text{in } L^{\frac{s}{s-N+1}}(\Omega)^N, \ \text{ or} \\ u_n \to u & \text{in } BMO(\Omega)^N, \ \text{ if } s = N-1 \geq 2 \end{array}\right\} \Rightarrow \mathrm{Det}\,(Du_n) \rightharpoonup \mathrm{Det}\,(Du) \ \text{ in } \mathscr{D}'(\Omega). \tag{3.72}$$

In view of the div-curl results of Section 2, we will prove a weak continuity result for the Jacobian assuming that $\mathrm{Det}\,(Du_n)$ converges weakly in $W^{-1,1}(\Omega)$ and that u_n converges slightly better than weakly in $W^{1,N-1}(\Omega)$.

Theorem 3.8. *Let Ω be a bounded open set of $\mathbb{R}^N$, with $N \geq 2$. Consider a sequence u_n in $W^{1,N}(\Omega)^M$ satisfying*

$$\mathrm{Det}\,(Du_n) \rightharpoonup \mu \ \textit{ in } W^{-1,1}(\Omega). \tag{3.73}$$

We have the following alternative:

- *Assume that there exists $s > N-1$ such that*

$$u_n \rightharpoonup u \quad \textit{in } W^{1,s}(\Omega)^N. \tag{3.74}$$

  If $u \in W^{1,N}(\Omega)^N$, then $\mu = \mathrm{Det}\,(Du)$.
  Otherwise, μ is given by the weak formulation

$$\begin{cases} \forall\, B(x_0, R) \Subset \Omega,\ \forall\, \varphi \in W^{1,\infty}(0,\infty),\ \textit{with } \operatorname{supp}\varphi \subset [0,R], \\[2mm] \langle \mu, \psi \rangle = -\displaystyle\int_\Omega \left(\sum_{j=1}^N \mathrm{cof}\,(Du)_{1j}\, \partial_j \psi\, u^1 \right) dx, \quad \textit{where } \psi(x) := \varphi(|x - x_0|). \end{cases} \tag{3.75}$$

- *Or else, assume that*

$$u_n \rightharpoonup u \quad \begin{cases} \textit{in } W^{1,N-1}(\Omega)^N, & \textit{if } N > 2 \\ \textit{in } BV(\Omega)^N *, & \textit{if } N = 2, \end{cases} \tag{3.76}$$

  and that ∇u_n^1 belongs to $L^{N-1,1}(\Omega)^N$ with the equi-integrability condition

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \|\nabla u_n^1\|_{L^{N-1,1}(E)^{M\times N}} \leq \varepsilon,\ \forall\, n \in \mathbb{N},\ \forall\, E \textit{ measurable} \subset \Omega,\ |E| < \delta. \tag{3.77}$$

If $u \in W^{1,N}(\Omega)^N$, *then* $\mu = \mathrm{Det}\,(Du)$.
Otherwise, μ *is given by the weak formulation*

$$\begin{cases} \forall\, B(x_0,R) \Subset \Omega,\ \forall\, \varphi \in W^{1,\infty}(0,\infty),\ \textit{with}\ \mathrm{supp}\,\varphi \subset [0,R],\ \textit{such that} \\ \exists\, U\,\textit{closed set of}\ [0,R],\ \textit{with}\ u(x_0+ry) \in C^0(U; X^{1,N-1}(S_{N-1}))^M,\ \mathrm{supp}\,(\varphi') \subset U, \\ \displaystyle \langle \mu, \psi \rangle = -\int_\Omega \left(\sum_{j=1}^N \mathrm{cof}\,(Du)_{1j}\, \partial_j \psi\, u^1 \right) dx, \quad \textit{where}\ \psi(x) := \varphi(|x-x_0|), \end{cases} \tag{3.78}$$

and $X^{1,N-1}(S_{N-1})$ *is the space defined by* (2.53).

Remark 3.9. The first case of Theorem 3.8 provides an improvement of the weak continuity (3.72) with $s > N-1$, given in [8]. Indeed, if a sequence u_n converges weakly in $W^{1,s}\Omega)$ and strongly in $L^{\frac{s}{s-N+1}}(\Omega)^N$, then by the classical weak convergence of the Jacobian (see, *e.g.*, [22], Corollary 2.8) $\mathrm{cof}\,(Du_n)$ converges to $\mathrm{cof}\,(Du)$ in $L^{\frac{s}{N-1}}(\Omega)^{N\times N}$. Hence, since the exponents $\frac{s}{s-N+1}$ and $\frac{s}{N-1}$ are conjugate, we obtain the weak convergence

$$\sum_{j=1}^N u_n^1\, \mathrm{cof}\,(Du_n)_{1j} \rightharpoonup \sum_{j=1}^N u^1\, \mathrm{cof}\,(Du)_{1j} \quad \text{in } L^1(\Omega),$$

which thus implies assumption (3.73). More generally, a sufficient condition to ensure assumption (3.73) is that

$$\sum_{j=1}^N u_n^1\, \mathrm{cof}\,(Du_n)_{1j} \quad \text{is equi-integrable in } L^1(\Omega).$$

The second case of Theorem 3.8 proposes an alternative to the delicate weak continuity (3.72) with $s := N-1$, obtained in [8] (Theorem 1), in which the strong convergence of u_n in $L^\infty(\Omega)^N$ is replaced by the weak convergence of $\mathrm{Det}\,(Du_n)$ in $W^{-1,1}(\Omega)$ combined with the equi-integrability of ∇u_n^1 in the Lorentz space $L^{N-1,1}(\Omega)^N$. There is no link between these two sets of assumptions.

Proof of Theorem 3.8. First of all and similarly to the proof of Theorem 2.1, the regularity assumption $u \in W^{1,N}(\Omega)^N$ implies that the weak formulations (3.75) and (3.78) lead to the equality $\mu = \mathrm{Det}\,(Du)$. It thus remains to treat the general case for $s > N-1$ and $s = N-1$.

The case: $s > N-1$.

In view of the classical weak continuity (3.71) we may restrict ourselves to the case $s \le \frac{N^2}{N+1}$. Define $p := \frac{s}{N-1}$ and $q := s$. Let us check that the sequences of vector-valued functions $\eta_n := \nabla u_n^1$ and σ_n defined by

$$\sigma_n^i := \left[\mathrm{cof}\,(Du_n)\right]_{1i}, \quad \text{for } 1 \le i \le N. \tag{3.79}$$

satisfy the assumptions of Theorem 2.1:

First, σ_n and η_n satisfy condition (2.2) with exponent $N' \in [p, q']$ since

$$p = \frac{s}{N-1} \le \frac{N^2}{N^2-1} \le N' \le \frac{N^2}{N^2-N-1} = \left(\frac{N^2}{N+1}\right)' \le s' = q'.$$

while $p, q > 1$ satisfy the inequality

$$\frac{1}{p} + \frac{1}{q} = \frac{N}{s} < 1 + \frac{1}{N-1}.$$

Next, σ_n is divergence free and by the classical weak convergence of the Jacobian (see [22], Corollary 2.8) σ_n converges weakly in $L^p(\Omega)^N$ (since $p = \frac{s}{N-1}$) to the function $\sigma = (\sigma^1, \ldots, \sigma^N)$ given by

$$\sigma^i = \big[\mathrm{cof}\,(Du)\big]_{1i}, \quad \text{for } 1 \leq i \leq N. \tag{3.80}$$

Moreover, η_n is curl free and converges weakly to ∇u^1 in $L^q(\Omega)^N$. Therefore, taking into account convergence (3.73), Theorem 2.1 through (2.8) yields the desired limit formulation (3.75).

The case: $p := 1$ *and* $q := N-1$.

Let us check that the sequences of vector-valued functions σ_n defined by (3.79) and $\eta_n := \nabla u_n^1$ satisfy the assumptions of Theorem 2.11:

As in the previous case σ_n and η_n satisfy condition (2.2) with $s = N'$. Next, σ_n is divergence free in Ω, and by the classical weak convergence of the Jacobian (see, [22], Corollary 2.8) σ_n converges weakly-$*$ in $\mathscr{M}(\Omega)^N$ to the function σ defined by (3.80). Moreover, η_n is curl free and converges weakly to ∇u^1 in $L^{N-1}(\Omega)^N$. Therefore, taking into account conditions (3.73) and (3.77) Theorem 2.11 (see also Remark 2.12) applies and leads to the limit formulation (3.78), which concludes the proof. $\square$

The following example shows that Theorem 3.8 does not hold if we just assume that u_n converges weakly in $W^{1,N-1}(\Omega)$ for $N > 2$, or weakly-$*$ in $BV(\Omega)$ if $N = 2$. We refer to [23] (Theorem 1) to an alternative counterexample with the critical space $W^{1,N^2/(N+1)}(\Omega)$ related to the weak continuity (3.71).

Example 3.10. Let $N \geq 2$, and let Ω be the cylinder $B_1' \times (0,1)$, where B_1' is the unit ball of $\mathbb{R}^{N-1}$. The points of Ω are denoted by (x', x_N). We also use x' to denote a point of Ω whose last coordinate is zero.

Define in cylindrical coordinates the vector-valued function u_n in Ω by

$$u_n(x) := (1-r)^n\,(nx', x_N), \quad \text{for } x \in \Omega, \quad \text{where } r := |x'|.$$

Then, we have

$$\begin{aligned} Du_n' &= -\,n\,(1-r)^{n-1}\left[n\,\frac{x'\otimes x'}{r} - (1-r)\,I_{N-1}\right], \\ \nabla u_n^N &= (1-r)^{n-1}\left[-\,n\,\frac{x_N\,x'}{r} + (1-r)\,e_N\right]. \end{aligned}$$

Therefore, as a consequence of (2.81) we get that

$$\int_\Omega |Du_n|^{N-1}\,dx \leq C\,n^{2N-2}\int_0^1 r^{2N-3}(1-r)^{(n-1)(N-1)}\,dr.$$

$$+\,C\,n^{N-1}\int_0^1 r^{N-2}(1-r)^{(n-1)(N-1)}\,dr + C \leq C.$$

This combined with the convergence of u_n to zero a.e. in Ω, implies that

$$\begin{cases} u_n \overset{*}{\rightharpoonup} 0 \ \text{ in } BV(\Omega)^N & \text{if } N = 2 \\ u_n \rightharpoonup 0 \ \text{ in } W^{1,N-1}(\Omega)^N & \text{if } N > 2. \end{cases}$$

On the other hand, it is easy to check that

$$\mathrm{Det}\,(Du_n) = \det\,(Du_n) = n^{N-1}\,(1-r)^{nN} - n^N\,r\,(1-r)^{nN-1}r.$$

Hence, again using (2.81) we conclude that

$$\operatorname{Det}(Du_n) \overset{*}{\rightharpoonup} |S_{N-2}| \lim_{n\to\infty} \left[\int_0^1 \left(n^{N-1} r^{N-2} (1-r)^{nN} - n^N r^{N-1} (1-r)^{nN-1} \right) dr \right] \left(\delta_{\{x'=0\}} \otimes 1 \right)$$
$$= |S_{N-2}| \, \frac{(N-2)!}{N^N} \left(\delta_{\{x'=0\}} \otimes 1 \right) \neq 0 \quad (\text{with } |S_0| := 1).$$

Finally, note that

$$\lim_{n\to\infty} \|u_n\|_{L^\infty(\Omega)} = \lim_{n\to\infty} \left[\max_{r\in[0,1]} \left\{ n\, r\, (1-r)^n \right\} \right] = \frac{1}{e} > 0,$$

which also illustrates the sharpness of the weak continuity result (3.72) in [8].

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